ELEMENTARY ALGEBRA

EDGERTON AND CARPENTER

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ELEMENTARY ALGEBRA

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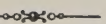

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PREFACE

In this book algebra is treated as the *science of the equation*. Beginning with an introductory chapter on algebraic notation and the formula, Part I leads by a series of six carefully graded, logical divisions through the simple equation. By equally logical steps Part II takes the pupil through the quadratic equation in six more chapters. Part III has two chapters devoted to the study of numerical trigonometry.

To emphasize further the importance of the equation and to apply it wherever possible, each chapter contains a series of equation-problems involving the use of equations in connection with the topics treated in that chapter.

But perhaps the *most important feature* of the book is the one which has met such favor in the case of the authors' First and Second Courses in the New Mathematics. Not only does each chapter grow logically out of its predecessor, but each topic in every chapter follows logically what precedes and leads up to what follows. Furthermore each chapter is introduced by a brief statement linking it with the subject matter that has preceded.

The authors have aimed to make the *introductions* interesting and appropriate. Not only does the general Introduction bridge the gap between arithmetic and algebra, but the introductory chapter to Part I leads through a treatment of the formula to types of simple equations arithmetically solved. In the same way the first chapter in Part II, through products and factors, leads to the factoring solution of the quadratic. Thus each introductory chapter gives a simple conspectus of the aim of each Part.

The treatment of *graphs* is unique. Graphs are of two

kinds: illustrative arithmetical graphs, like the statistical graphs, and functional, algebraic graphs, like those of the formula and the equation. The former are disposed of in the general Introduction, where they form a link between the mathematics of the previous grades and the present algebra. The latter are inserted in the text in connection with the topic they exemplify, instead of being grouped in a separate chapter which breaks the sequence of the book.

Tests of all kinds abound throughout the book. Each chapter has at least two, and these are chosen (as selection, multiple-choice, completion, timed, true-false, matching, and so on) for their adaptability to the subject matter treated.

Geometry also finds an important place. Lines, angles, triangles, polygons, and circles are all treated in their algebraic applications at appropriate points.

The *illustrations* are meant to add life and interest to the work. In this pictorial age pupils want their work illustrated wherever possible. This is done by pictures, by graphs, and by blocked types of specific solutions.

Illustrative examples supplement the simple, orderly explanations by which each topic is introduced.

Cumulative reviews with each chapter apply in well graded problems all the principles previously treated. These reviews are supplemented by an equation review in each chapter.

The *examples and problems* are a special feature. There are more than in any other one book on the subject.

The text has profited by suggestions from teachers throughout the country. In particular, the authors acknowledge with gratitude the helpful advice of Mr. Joseph P. McCormack, Chairman of the Mathematics Department, Theodore Roosevelt High School, New York City.

E. I. E.
P. A. C.

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ELEMENTARY ALGEBRA

INTRODUCTION. STATISTICAL GRAPHS

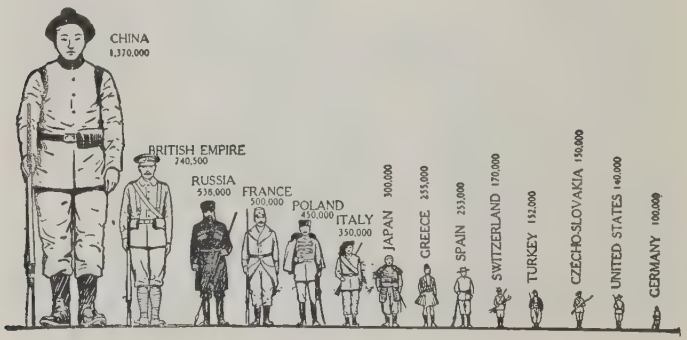
We all like the "movies." Pictures tell a story more vividly and more quickly than words. They often enable us to see at a glance things that it might take many words

A Graphic Picture

to describe. For instance, it takes a whole sentence to tell how many times as large one dog is as another. But a picture of the two dogs together not only shows this at a glance, but gives us a better idea of the difference.

Graphs. — The same is true of *graphs*. A graph is a picture of the relation of two or more facts. We frequently fail to grasp all the relations from a set of figures in tabulated form, but when these are pictured by means of a graph, we see at once the true relations. Besides, we find graphs frequently used in magazines, daily papers, textbooks, scientific journals, and reports. So we should learn how to interpret them, that is, tell the meaning of the relations between the numbers graphed (pictured). As we go on in this book we shall learn how to graph formulas, equations, and many other things as well as the statistics in this Introduction.

Picture Graphs. — Any diagram which shows the relative size or magnitude of two or more quantities or the rate at which one quantity changes is called a *graph*. Many devices are used for this purpose. The following is from the *Literary Digest* and shows a *pictorial* graph comparing the size of the world's largest armies.



The Bar Graph. — Another common form of graph consists of straight, heavy lines or bars of equal width, either vertical or horizontal, whose lengths represent the relative magnitude of the quantities compared. The graphs shown on the next page are typical bar graphs.

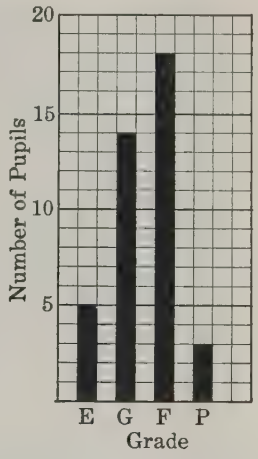
The bar graph we see here pictures the marks obtained in a certain test in mathematics. The figures at the left show the number of pupils. The letters at the bottom show the grades received. This is a *vertical* bar graph.

Interpretation.

This graph may be interpreted by studying the answers to the following questions.

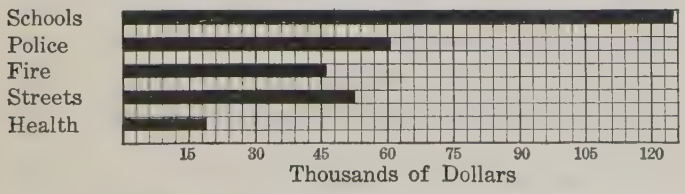
How many pupils obtained

(a) E (90 to 100)?
(b) G (80 to 90)?
(c) F (70 to 80)?
(d) P (Failed)?
(e) How many took the test?
(f) What per cent failed?



When bar graphs represent large numbers, they are not supposed to be absolutely accurate.

Below is a *horizontal* bar graph, to illustrate the following costs. In a certain city the cost of the public schools was \$125,350; police \$60,100; fire department \$45,800; streets and parks \$52,600; public health \$18,250. To picture the



relation of these costs by the use of bar graphs, we draw a heavy vertical line near the left side of our graph sheet. This is a specially ruled paper, very convenient in making graphs. We make all the bars start evenly at that line. The longest

bar must represent \$125,350. The longest that we can make it with the paper that we are using is 9 large squares. Each large square then must represent at least $\frac{1}{9}$ of \$125,350, which is nearly \$14,000. Let us then make one large square represent \$15,000 and then each small square here will represent

POLICEMEN ON PARADE

From the graph on the preceding page you can tell at a glance how police expenses compare with school expenses.

\$3000 if we use paper like that above, that has 25 small squares in one large square. \$125,350 ÷ 15,000 = 8 large squares, with \$5350 remaining. \$5350 ÷ 3000 = 1.78^{+} small squares.

Hence the bar, which represents \$125,350, has a length equal to 8 large squares and a little more than $1\frac{3}{4}$ small squares. In a similar way we divide \$60,100 by 15,000 to find the bar representing police expenses; and so on for the other departments.

Interpreting the Graph on page xi.

(a) Which department costs the most?

(b) It costs about twice as much as the . . .

(c) Compare the costs of police, fire, and street departments (approximately).

Suggestions for Making Bar Graphs.

1. Use a sharp pencil, and a ruler.

2. Measure accurately.

3. Choose a unit fitted to the paper that you have. The unit is the number that the smallest square represents. In the preceding graph it was 3000.

4. In a *horizontal* bar graph, the scale begins at the left and goes to the right.

5. In a *vertical* bar graph, the scale begins at the bottom and goes upward.

6. As we have learned, in bar graphs the lengths of the bars are proportional to the magnitudes of the quantities represented. It is necessary, therefore, to choose a unit such that there will be room on the paper for the longest bar. Squared paper can be used to best advantage in doing this work, especially if it is ruled 10 squares to the inch. In all graph work try to arrange your chart on the paper in such a way as to have equal width of margin of the paper at the sides, and equal width of margin at the top and bottom of the graph.

Exercise

Draw bar graphs to represent each of the following. Interpret each graph.

1. The number of pupils present in a home-room for 5 successive days was 32, 30, 31, 32, 27.

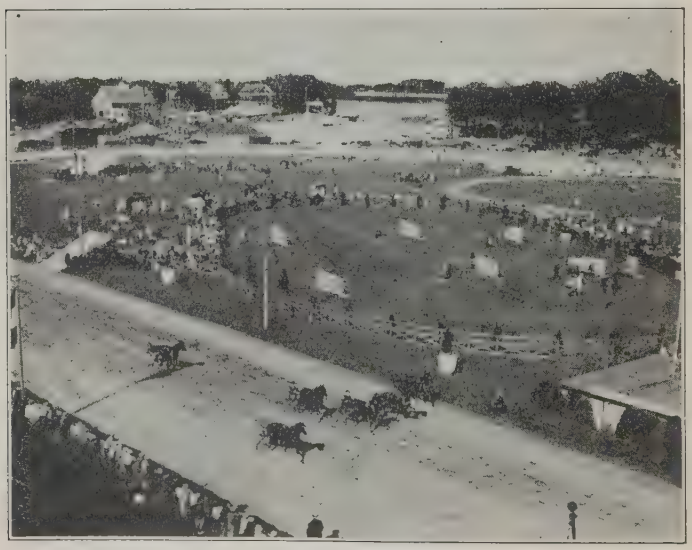
2. Ten pupils of an algebra class receiving the following marks in an examination: 100, 60, 90, 80, 95, 93, 80, 78, 92, 85.

3. The production of cotton in the U. S. to the nearest million bales.

1921 — 8	1923 — 10	1925 — 16	1927 — 13
1922 — 10	1924 — 14	1926 — 18	1928 — 14

4. The number of people in the U. S. with no education at all (called illiterate) as given by the U. S. Census reports in millions.

1880 — 6.2 1890 — 6.3 1900 — 6.2 1910 — 5.5 1920 — 4.9



A Country Fair

Do you think this was Tuesday or Thursday?

5. The attendance at a country fair was

Tue. 1500 Wed. 3000 Thur. 3500 Fri. 1800

6. The causes of deaths in N. Y. City in a recent year and the figures:

Cancer	7033	Pneumonia . . .	10521
Children's diseases .	1002	Tuberculosis . . .	4828
Heart	15784	Violence	5987
Kidney	4361	Other	2099

Use vertical bars. 1 inch might represent 2000.

7. A family's ice bill for 6 months beginning with June was

Month . . .	1	2	3	4	5	6
Cost . . .	\$1.25	\$2.50	\$2.25	\$1.75	\$1.50	\$1.00

8. The distances that different kinds of trucks can travel on \$1 expenditure has been reported as follows:

5-ton horse . .	1.6 miles	$3\frac{1}{2}$-ton gas . .	2.4 miles
5-ton gas . .	1.8 miles	$3\frac{1}{2}$-ton electric .	2.75 miles
5-ton electric . .	2.3 miles	2-ton horse . .	2.93 miles
$3\frac{1}{2}$-ton horse . .	2.2 miles	2-ton gas . .	2.6 miles
		2-ton electric .	3.3 miles

9. Expenditures (cost of government) of N. Y. State, in millions.

1921 — 136	1923 — 136	1925 — 167	1927 — 189
1922 — 130	1924 — 151	1926 — 178	1928 — 218

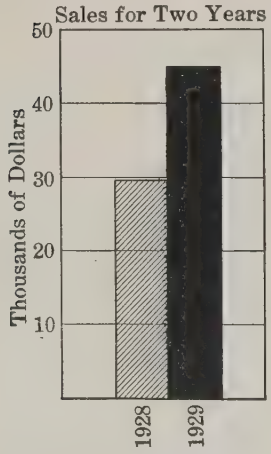
10. Debt in N. Y. State, in millions.

1921 — 268	1923 — 264	1925 — 318	1927 — 341
1922 — 267	1924 — 308	1926 — 315	1928 — 360

11. An object, falling freely, falls the following number of feet in the respective number of seconds.

Time (Sec.) .	0	1	2	3	4	5
Falls (Feet) .	0	16	64	144	256	400

The Double (Dual-) Bar Graph is used to compare two similar items for two respective years or two related items for the same year.



This graph shows the relation between the sales of the same salesman for two years, — the light bar representing the sales of one year and the shaded bar those of the other year.

Interpretation

How many thousand dollars' worth of goods did he sell each year?

His 1929 sales were how much more than his 1928 sales?

Do you think such a salesman should keep his job? Why?

Exercise

Make dual-bar graphs to represent:

1. The sales of one salesman whose sales for two different years were \$10,000 and \$14,000 respectively. The corresponding selling expenses for the two years were \$1400 and \$1625. Make two pairs of dual bars.

2. The rough materials used in manufacturing in a certain year in millions of dollars were exports 1.9, imports 1.8. (From and to the U. S.)

3. The capital and production in millions of dollars were: 1st year C, 78, P, 86; 2d year (40 years later) C, 5000, P, 1500.

4. The gasoline tax in the U. S. for 1927 and 1928 was 259 millions and 280 millions of dollars respectively. The

motor vehicle registration (all licenses) fees in the U. S. for 1927 and 1928 were 301 millions and 325 millions of dollars. The total cost of owning and operating automobiles in the U. S. for 1927 and 1928 was 770 millions and 785 millions of dollars. Make two pairs of triple bars.

A Gasoline Plant

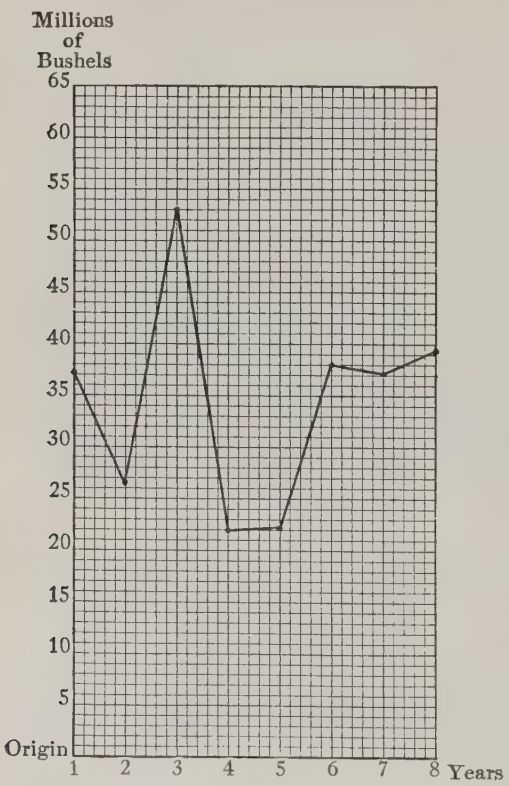
Constructing the Broken-line Graph. — The potato crop of the state of New York in million bushels for eight consecutive years was:

37.1, 26.6, 53.2, 22, 22.4, 38, 37.2, 39.5

For the purpose of constructing a broken-line graph, use paper ruled in squares (preferably 10 to the inch) like that described under bar-graphs. This graph paper is sometimes called "cross-section" or "coördinate" paper. Two base lines at right angles to each other are drawn, one horizontal, the other vertical. These are called *axes*, and their intersection is called the *origin*. On the horizontal axis, we begin at the origin and mark off to the right equal spaces to represent equal intervals of time. In this case one large square on the cross-section paper represents one year.

We then begin at the origin and mark off on the vertical

axis upward from the origin equal spaces to represent equal numbers of bushels. We shall let one large square on the paper represent 5 million bushels. We are now ready to locate the various points on the graph.



On the vertical line directly above each specified year on the horizontal axis, place a point exactly opposite the correct number of bushels for that year as shown on the vertical axis. After locating all such points, join them by straight lines in regular order.

Interpretation.

We are now able to read at once from the graph the answers to such questions as these:

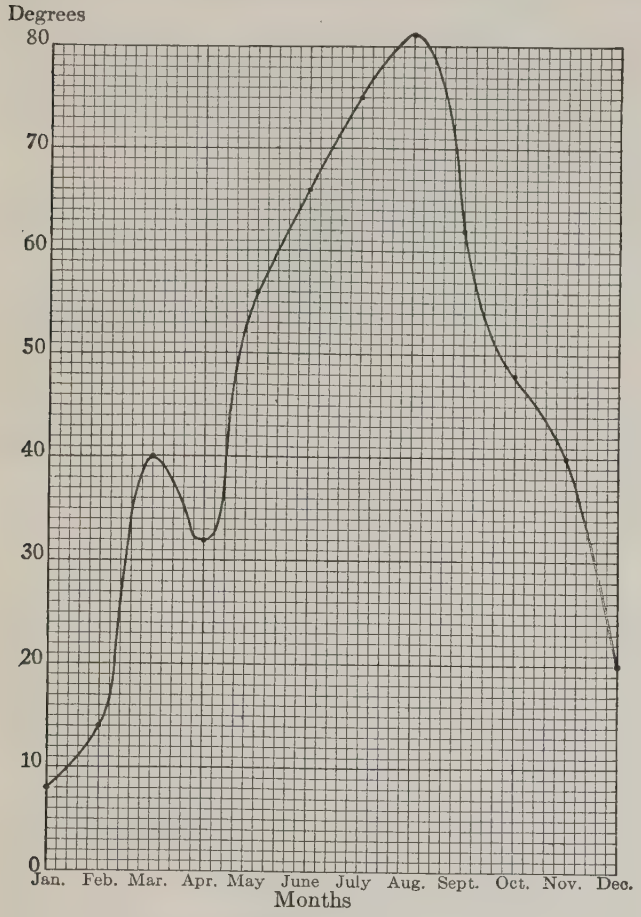
In which year did New York State produce the most potatoes?

What was the yield that year?

In which two consecutive years was the crop very poor?

What was the yield in each of those years?

Was there much variation during the last three years?
Was the yield in the last three years high, low, or average?



Constructing the Curved-line Graph. — The curved-line graph is constructed like the broken-line graph except that after the points are located they are joined by a smooth curve instead of straight-line segments.

The average temperature for each month of a certain year was recorded by the weather bureau as follows:

Jan. 8°, Feb. 14°, March 40°, April 32°, May 56°, June 66°, July 75°, Aug. 81°, Sept. 62°, Oct. 48°, Nov. 40°, Dec. 20°.

The curved-line graph on the preceding page shows the temperature changes through the year.

Interpretation.

Answer the following questions by inspection of the graph:

1. (**a**) Which month was warmest? (**b**) Which was coldest?

2. What two months had the same average temperature?

3. Which months had average temperatures below freezing (32°)?

4. What peculiar thing do you notice about spring temperatures?

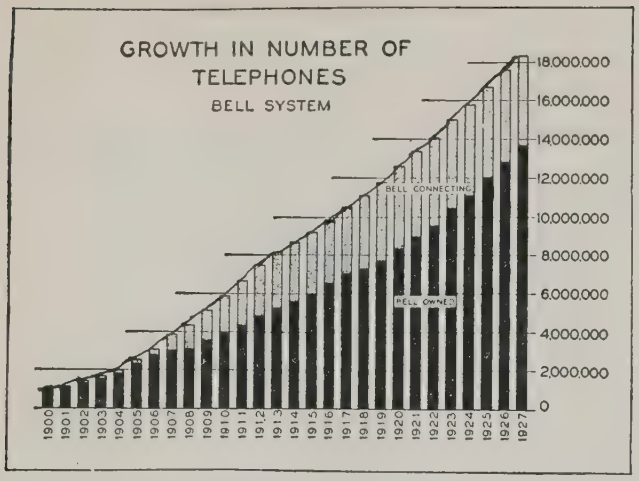
5. Was the fall of temperature after August as rapid as the rise before August?

The curved-line graph should be used when the subject of the graph changes gradually from one value to another. The broken-line graph should be used when the subject changes abruptly. For example, the temperature did not jump but changed gradually from one value to another. Hence the curved-line graph is used. On the other hand, the potato crop did not change gradually from 37.1 million the first year to 26.6 million bushels the second year. Hence the broken-line graph was used there.

In the following graph if the tops of the bars are connected we have a curved line graph.

(**a**) Between which two years is the curve nearly level?

(**b**) Why do you suppose that the curve is not as steep from 1913 to 1918 as it is from 1918 to 1927?



(c) The number of phones in 1927 is about . . . times as many as in 1900.

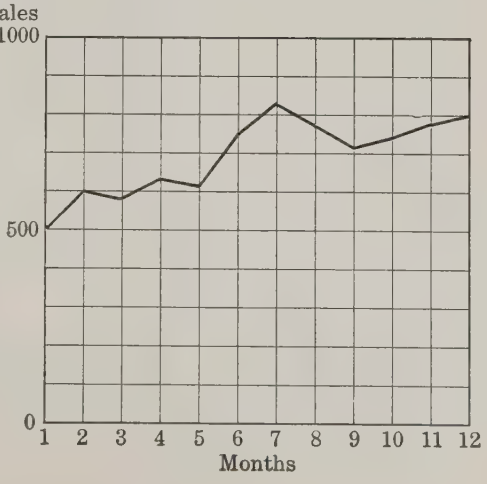
(d) The number has about . . . since 1916.

Tell from the adjoining sales graph

(a) In which months the sales decreased.

(b) The most sales were made in the . . . month.

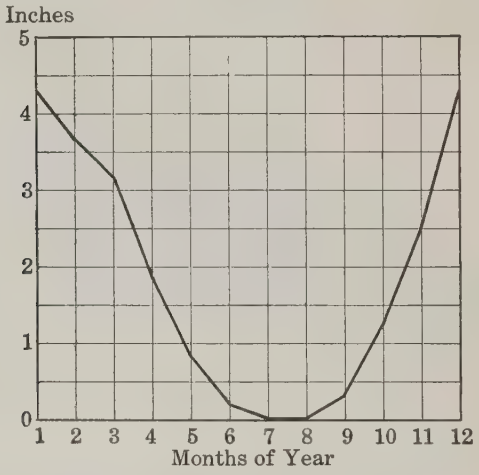
(c) The sales were lowest in the . . . month.



From the accompanying rainfall graph tell

(a) Which is the dry season?

(b) Which is the wet season?



(c) In what months was there no rainfall?

(d) Which months had the greatest rainfall?

Exercises

Draw either broken-line or curved-line graphs, whichever seem the better adapted to the problems, for each of the following statistical tables. Be sure to follow closely the suggestions given above, and then discuss and interpret each graph as directed by the teacher:

1. A boy made the following final marks in mathematics during six successive terms: 94, 80, 85, 88, 90, 92. Make a graph to show the above.

2. The average monthly temperatures of New York City in a recent year were 31°, 31°, 38°, 49°, 61°, 69°, 74°, 73°, 67°, 56°, 44°, 35°.

Which month was the hottest?

3. The average monthly rainfall (precipitation) of the above city in inches from January was: 3.8, 3.7, 4.1, 3.3, 3.2, 3.3, 4.5, 4.5, 3.6, 3.7, 3.4, 3.4.

Some of the Weather Man's Instruments

4. The cost of gas in a home beginning with Nov.

Month . . .	1	2	3	4	5	6
Cost . . .	$2.07	$4.14	$4.37	$4.37	$2.99	$.46
Month . . .	7	8	9	10	11	12
Cost . . .	$1.15	$.69	0	0	$1.38	$3.91

The home was closed for part of April.
Why was there no gas bill in July and August?

5. The population of the United States by decades from 1790 to the nearest million was 4, 5, 7, 10, 13, 17, 23, 31, 39, 50,.63, 76, 92, and 106.

Tell in which decade the population increased the fastest.

6. Approximate population of the British Isles by decades from 1801 in millions was 9, 10, 12, 14, 16, 18, 20, 23, 26, 29, 32, 36, 38.

7. The temperatures of a pneumonia patient were recorded as follows:

12 Noon . .	103.1 degrees	7 P.M. . .	104.4 degrees
1 P.M. . .	103.2 "	8 P.M. . .	104. "
2 P.M. . .	103.6 "	9 P.M. . .	103.7 "
3 P.M. . .	104. "	10 P.M. . .	103.2 "
4 P.M. . .	104.2 "	11 P.M. . .	103. "
5 P.M. . .	104.8 "	12 Midnight	102.8 "
6 P.M. . .	104.6 "		

(a) When was the temperature highest? Lowest?

8. The following table shows the average weight of boys from 6 to 16.

Age	6	8	10	12	14	16
Weight (lbs.)	45	54	65	77	95	121

From your graph determine the approximate weight of a boy of 15. Of 11.

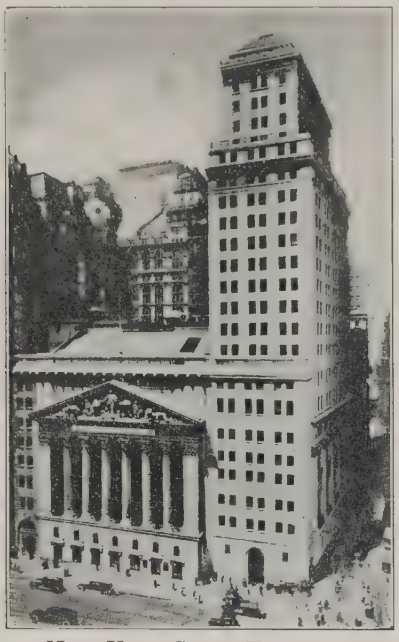
9. Table of ages and weights of girls.

Age	6	8	10	12	14	16
Weight (lbs.)	43	52	62	78	98	114

Compare and discuss the above graph with that for example 8.

10. Average price per dozen of eggs in United States for nine successive years.

33¢, 33.2¢, 30.5¢, 36.3¢, 46.1¢, 53.6¢, 60.2¢, 63.6¢, 61.1¢.



New York Stock Exchange

11. The high value of the same 25 industrial stocks on the N. Y. Stock Exchange as reported by the *N. Y. Times* was

1916	1917	1918	1919	1920	1921
$119.30	$99.74	$91.55	$138.12	$129.83	$90.60
1922	**1923**	**1924**	**1925**	**1926**	**1927**
$116.24	$118.44	$135.11	$185.36	$186.03	$247.48

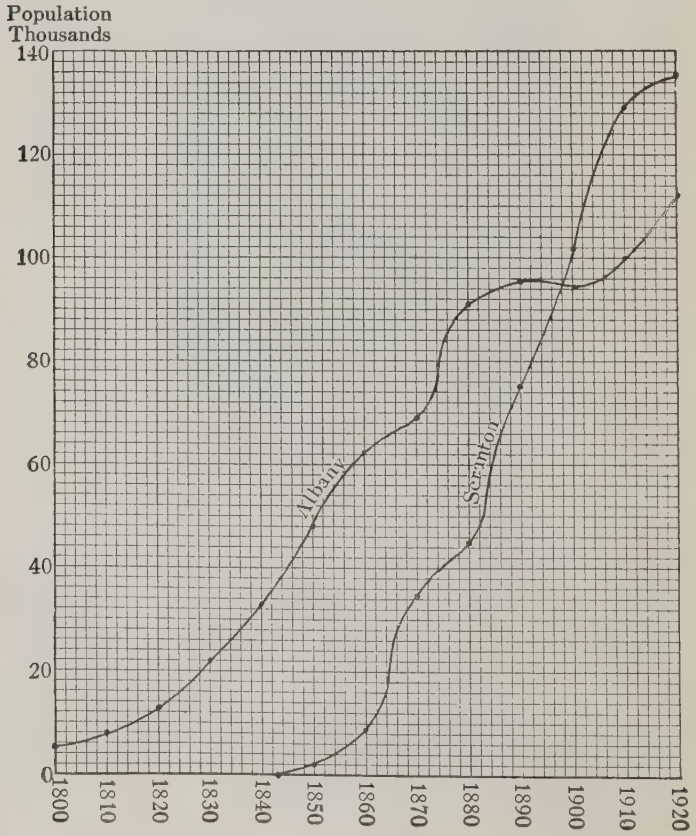
12. The average monthly temperatures of Dallas, Texas from January for twelve months were:

45°, 47°, 56°, 65°, 73°, 81°, 84°, 83°, 77°, 66°, 55°, 47°.

13. The average monthly temperatures of St. Paul.

13°, 16°, 29°, 46°, 58°, 67°, 72°, 69°, 61°, 49°, 32°, 19°.

Comparison of Graphs. — It is often advantageous to draw two similar graphs on the same axes for purposes of comparison. The graphs which follow show the growth of population of two American cities, Albany, N. Y., and Scranton, Pa.



Interpretation.

Study the preceding graph and answer the following questions:

Which is the older city?

Which has had the more rapid growth?

State the approximate population of each in 1920. Compare their rates of growth during the last ten years. During what ten years did Albany make no gain? When were the two cities equal in population? What was their population at that time?

Exercises

1. Draw graphs on the same axes to show the relative size and growth of Philadelphia and Chicago. The tables give the population in thousands.

	1790	1800	1810	1820	1830	1840	1850
P.	28	41	54	64	80	94	121
C.						4	28
	1860	**1870**	**1880**	**1890**	**1900**	**1910**	**1920**
P.	566	674	847	1047	1294	1509	1823
C.	109	307	503	1106	1699	2185	2702

About when were their populations equal?

2. Represent the outward messages from a certain number of selected stations of the American Telephone Company for the twelve months of two successive years from the following data:

	Jan.	Feb.	March	Apr.	May	June
1st yr.	15,500	15,900	15,400	15,800	15,800	15,600
2d yr.	16,300	16,400	15,600	16,100	16,100	16,150
	July	Aug.	Sept.	Oct.	Nov.	Dec.
1st yr.	14,900	14,500	15,700	16,000	16,200	16,000
2d yr.	15,500	15,700	15,200	15,500	16,500	16,500

3. The following table shows the earnings and spendings of a boy from April to October inclusive during a certain year. Graph them.

	Apr.	May	June	July	Aug.	Sept.	Oct.
Earnings	$2.00	$3.20	$4.80	$10.40	$12.00	$4.40	$5.20
Spendings	3.20	3.60	2.40	2.00	1.60	8.40	4.00

(a) When were they nearly the same?

(b) Why do you think the earnings were so much more in July and August?

4. Construct on the same chart the weekly sales of two salesgirls from the following data:

Salesgirls	1st Wk.	2d Wk.	3d Wk.	4th Wk.	5th Wk.	6th Wk.	7th Wk.
1 . .	$102	$ 40	$186	$160	$115	$120	$90
2 . .	70	126	130	160	255	220	86

What do you note concerning the respective sales of these two salesgirls during the fourth week? The seventh week?

What use do you think the manager of the business could make of such curves continuing over a period of years?

Do you think of any other use that might be made of the curves representing the sales of different salesgirls?

5. The monthly earnings and expenses of a company were reported as follows:

	Jan.	Feb.	March	April	May	June
Earnings	$73,000	$63,000	$77,000	$76,000	$85,000	$91,000
Expenses	47,000	51,000	53,000	51,000	60,000	63,000
	July	Aug.	Sept.	Oct.	Nov.	Dec.
Earnings	$105,000	$109,000	$88,000	$82,000	$75,000	$70,000
Expenses	73,000	77,000	74,000	77,000	60,000	55,000

Make two curves, one to represent earnings, the other expenses.

6. Compare the following sales and expenses of selling for the 5 years by the use of curved-line graphs:

	1924	1925	1926	1927	1928
Sales . . .	$125,000	$200,000	$315,000	$290,000	$350,000
Selling Expenses .	12,000	17,500	26,000	24,000	26,250

7. The rainfall in inches per month was reported by the weather bureau from each of the cities mentioned below:

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Atlanta . . .	5.3	4.6	5.8	3.6	3.1	3.9	4.7	4.5	3.5	2.3	3.4	4.5
Kansas City . .	1.1	1.6	2.8	3.3	5.1	4.7	4.8	4.8	3.8	2.2	1.8	1.4

Make curve graphs of the rainfall. From your graph tell:

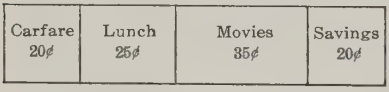
(a) Which city had the most rainfall from June to Sept. inclusive.

(b) Can you tell the chief crop produced near each of the above cities, and why it might grow well.

The Rectangular Distribution Graph pictures each item as a proportional part of the whole.

Example. A boy used a dollar as follows: carfare 20¢, lunch 25¢, movies 35¢, and he saved 20¢.

Solution: It is plain that 20 makes the most convenient unit. As a dollar contains 20 cents five times we take a rectangle of suitable width and 5 large squares long if graph paper is available. Let each square represent 20 cents and make the other items proportional.



From the graph you can easily see what part of the boy's dollar was used for each item.

Exercises

Make rectangular distribution graphs for the following:

1. A pupil spends 6 hours in school, 9 hours for sleep, 1 hour for study, 4 hours for play and work, and 4 hours for other uses.

Hint. Use one small square for each hour, making your graph 24 small squares long. It may be any convenient width.

2. The cost of a manufactured article.

Material 20¢ Labor 12¢ Overhead 6¢ Profit 2¢

3. A man earns $3200 a year. He spends 30% of it for board and room, 10% for clothes, 5% for amusement, $12\frac{1}{2}$% for luxuries, $6\frac{1}{4}$% for his church, and saves the balance. Show this with a rectangle graph. What fractional part of his income applies to each item?

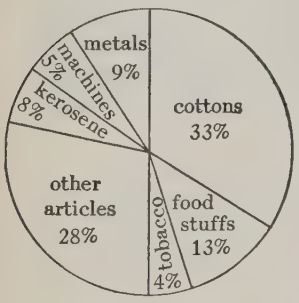
4. Suppose that you have 35 periods a week in your school. Suppose that you have 5 periods of English, 5 of Latin, 4 of History, 7 in Chemistry and 2 in Physical Training.

Make a rectangle 7 inches long (1 inch stands for 5 periods) and $\frac{3}{4}$ of an inch wide, then fill it in with these periods to the best of your ability.

5. The operating expenses of a small system of chain stores for a year were distributed as follows:

Buying and managing	16%	Fixed charges and upkeep	30%
Delivery charges	2%	Other expenses	10%
Selling expenses	42%		

The Circle Graph. — One of the common forms of distribution graphs is the *circle graph*. This consists of a circle which represents the whole of anything. Radii of the circle are drawn, dividing it into parts whose areas indicate the relative sizes of the numbers represented. The diagrams which follow are from the *Literary Digest* and are good examples of this kind of graph.

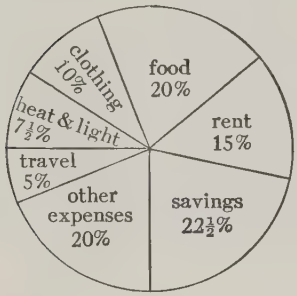


Construction of a Circle Graph. — The method of constructing a circle graph is shown by the following

Example. Mr. Brown's salary is $2000 per year. He spends $300 for rent, $400 for food, $200 for clothing, $150 for heat and light, $100 for travel and pleasure, and $400 for all other expenses. The remainder, $450, he puts in the bank. Construct a circle graph to show how he spends his salary.

300	54°
400	72°
200	36°
150	27°
100	18°
400	72°
450	81°
2000	360°

Solution: We see that he spent $\frac{300}{2000}$ for rent. The total number of degrees in the circle is 360, which represents his entire income. If, now, we take $\frac{300}{2000} \times 360°$, we obtain 54°, which is the angle showing what he spent for rent. In a similar manner we obtain the angles for the other items.



We now draw a circle of suitable size, and then at its center construct by means of a protractor, or otherwise, angles of the sizes indicated. Our graph is now complete. Your teacher will explain to you the protractor and its use if you are not familiar with this instrument.

Exercises

Show the distribution of each of the following by a circle graph:

1. Divide a circle to represent the following per cents: 30%, 20%, 40%, 10%.

2. George earns $3.00 a week. He spends $1.50 for lunches, 25¢ for his church, 30¢ for movies, 15¢ for candy, 50¢ for other things, and saves 30¢. Show this with a circle chart.

3. Apportion the sales of different branches of a business.

New York	30%	St. Louis . .	20%	Boston .	12%
Chicago .	10%	San Francisco	18%	Atlanta .	10%

4. In a certain high school there are
1780 Freshmen, 975 Sophmores, 595 Juniors, and 425 Seniors.

Make a rectangular or circular distribution graph of the above.

5. Make up a similar graph for your own school if the numbers can be obtained from the principal's office.

DAIRYMEN PREPARING MILK FOR SHIPMENT

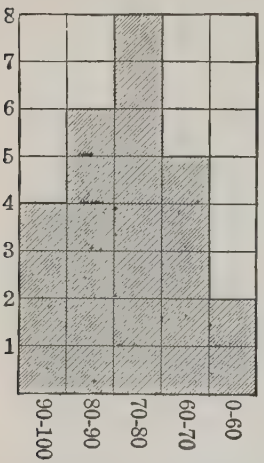
6. The payments out of the milk dollar by a Dairy Farm Products Co. Inc. in one year were about as follows:

To dairymen . . .	46¢	To shareholders	3¢
To labor	25¢	For materials, supplies,	
To railroads . . .	9¢	expenses of bottles, etc.	17¢

7. The gross revenue of the Bell Telephone System for one year was disposed of as follows:

Salaries and wages .	50%	Interest	19%
Materials, rent,		Surplus	6%
travel	20%	Taxes	5%

Frequency Distribution Graphs. — The accompanying graph shows how the marks for an examination were distributed in a class of twenty-five pupils. Such a distribution graph



Number of Pupils	Marks
4	90–100
6	80–90
8	70–80
5	60–70
2	50–60

may be applied to many kinds of data, such as the distribution of crops, population, rainfall, and so on. What per cent of the class got 90–100? 60–70? What is the ratio of the poorest group to the best group?

Exercises

Make a distributing graph of each of the following:

1. In a certain examination 3 pupils received marks between 90 and 100; 7 pupils between 80 and 90; 10 between 70 and 80; 4 between 60 and 70; and 1 at 55.

Hint: Use 1 small square to represent 1 pupil.

2. In a certain 8th-grade class the pupils ranked as follows: 1 at 96; 3 between 90 and 95; 5 between 85 and 90; 7 between 80 and 85; 8 between 75 and 80; 10 between 70 and 75; and 1 at 60.

3. The contributions of the various paying members of a certain church were as given on the next page. Use horizontal axis to represent money.

4	paid $7.00 per week.	13	paid $1.25 per week.
2	paid 5.50 per week.	30	paid 1.00 per week.
3	paid 5.00 per week.	11	paid .75 per week.
6	paid 3.50 per week.	14	paid .60 per week.
3	paid 2.75 per week.	40	paid .50 per week.
9	paid 2.00 per week.	45	paid .25 per week.
5	paid 1.75 per week.	25	paid .15 per week.
12	paid 1.50 per week.	30	paid .10 per week.

4. Jane's club lost money on its last entertainment. To make up the debt, members contributed as follows:

5	paid $2.00.	20	paid $.75.
10	paid 1.00.	40	paid .50.

Suggestions Concerning Graphs. — You should note the following suggestions concerning graphs:

The broken-line or curved-line graph can be used only to show the values of the *same quantity* recorded at *different times.*

To show the relative magnitude of *different quantities* at the *same time* use the circle or bar graph.

The graph should have an appropriate title.

The graph should be neither too small nor too large, and should be placed as nearly as possible in the center of the sheet.

In the case of a broken-line or curved-line graph the unit used should be plainly indicated on each axis.

Care should be exercised in choosing a suitable unit.

If the largest value to be represented is divided by the number of spaces available on the paper, the result will be the largest unit that can be used.

Miscellaneous Exercises

Select the type of graph you think best and graph the following:

1. The number of passenger automobiles produced in the U. S. in thousands.

1921 — 1400	1923 — 3500	1925 — 3800	1927 — 3100
1922 — 2400	1924 — 3200	1926 — 3900	1928 — 4000

2. The number of pupils present in a mathematics club on four successive monthly meetings was 20, 25, 15, 30.

3. The total popular vote for President in 1928.

Hoover 21,429,109. Smith 15,005,497.

4. The minimum (low) temperatures for 7 consecutive days including Christmas.

Chicago . . .	28°	20°	4°	18°	26°	36°	36°
New York . .	31°	35°	25°	20°	27°	34°	37°
Atlanta . . .	32°	38°	28°	30°	30°	38°	40°
San Francisco .	44°	44°	42°	44°	36°	48°	48°
Tampa . . .	48°	56°	50°	46°	42°	42°	42°

See if you can tell from your graph the probable trend of the weather for any of these places on the eighth day.

5. Out of a class of 30 taking an examination 3 pupils made 95%, 4 made 90%, 5 made 85%, 6 made 80%, 10 made 75%, and 2 made 60% or below. Show the above with a graph.

Hint: Number of pupils on vertical axis and per cents on the horizontal axis.

6. Out of 100 men aged 25, it is reported that when they reach the age of 65, 36 are dead, 1 is very wealthy, 4 are well off, 5 are self-supporting, and 54 are dependents.

Make a frequency polygon or rectangular distribution graph of the above.

7. Make a frequency polygon to show the following results in an examination:

4 stood between 90 and 100. 6 stood between 60 and 70.
7 stood between 80 and 90. 8 were below 60.
10 stood between 70 and 80.

Average, Median, and Mode. The simple *average* of a set of numbers is their sum divided by the number of numbers.

The *median* number of a set of numbers is the middle number of those numbers when they are arranged in order of size, from largest at the top to smallest at the bottom, provided there is an odd number of numbers. If there is an even number of numbers, it is the number halfway between the two middle numbers.

Example 1. The average weekly wage of 9 girls receiving respectively \$23, \$24, \$24.50, \$24.40, \$10, \$15, \$9.60, \$26, and \$20 is \$19.61. In fact no girl was receiving this wage.

To find the median wage these numbers should be arranged as follows:

\$26, \$24.50, \$24.40, \$24, **\$23,** \$20, \$15, \$10, \$9.60.

The median is \$23.

Example 2. Of two classes of 15 members each the tabulation on the next page represents the number of examples right in a certain time test. To find the median number of rights in each class.

Here you will note that if you count down from the top to the middle or the eighth number, that 11 are right in class 1 and 10 are right in class 2. We therefore say that 11 rights is the *median* for class 1 and 10 is the *median* for class 2. However, if you were to total the rights in class 1 and divide by 15, you would find that the average number of rights would be $12\frac{7}{15}$.

Class 1		Class 2	
Pupils	Rights	Pupils	Rights
a	21	a	20
b	18	b	17
c	17	c	16
d	15	d	14
e	14	e	13
f	13	f	12
g	12	g	11
h	11	h	10
i	11	i	10
j	11	j	10
k	10	k	9
l	10	l	9
m	9	m	8
n	9	n	8
o	6	o	7

The *modal number* or *mode* represents the commonest one of a series, as the commonest sized foot, the commonest weekly wage in a factory, or the commonest daily sales in a chain store. For example, the most common number of rights in class 1 is 11 and in class 2 is 10. The mode, however, is not always the same as the median.

Exercises

1. Find the average wage of 5 boys receiving \$16, \$10, \$12, \$18, and \$14.

2. Find the median wage of the above.

3. Find the average sales per week if Jones's sales for 5 weeks are \$500, \$700, \$600, \$800, and \$1000.

4. What is the median in example 3?

5. Find the average daily mileage made by a man in an automobile from New York to Los Angeles if his daily runs are: 320, 300, 260, 280, 277, 300, 350, 260, 280, 200, and 260 miles. What was his median day's run? His modal day's run?

6. In an examination in English

5 passed 90 to 100	17 passed 70 to 80
12 passed 80 to 90	6 failed below 70

About what do you think was the modal per cent?
How many do you suppose might get on well in Latin?

Need of Statistics. — Business to-day more than ever before depends on facts for its success. Such items as sales, deliveries, margins, clerk's sales, cash and account sales, overhead expenses, general and selling expenses, pay-roll calculations, and so on must be kept in concise form.

An Adding Machine

These facts are usually presented in tabulated form, which is probably the most convenient form to arrange them. You now have an opportunity to study a few of these forms. Let us see how well you remember your arithmetic. But in this work we must bear in mind that no adding machine or other mechanical device can take the place of training for accuracy.

Exercise

Copy and complete each of the following forms:

1. Subtract the second column from the first and place the result in the third column:

Selling Price of Goods Sold	Cost of Goods Sold	Margin (or Gross Profit)
$4000	$3200	
500	400	
5¢	3¢	

2. Find all totals:

CLERKS SALES FOR A WEEK

Clerk	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.	Total
1	$43.75	$94.75	$100.25	$75.64	$96.50	$125.65	
2	57.65	87.48	97.45	83.50	125.46	140.50	
3	75.52	67.82	92.25	87.75	130.67	150.58	
Total							

If they were all in the same department, which might appear to be the best clerk?

3. Complete:

Department	Cash Sales	Account Sales	Total Sales
1	$256.75	$124.43	
2	312.24	146.37	
3	175.32	142.13	
Total			

4. Calculate from the following pay roll how much each person earns during the week and their total earnings:

	Hours per Day						Total Hours	Rate per Hour	Total Earned
	M	T	W	T	F	S			
H. Jones	8	8	8	8	8	4	44	40¢	
F. Brown	8	8	8	8	6	3	41	50¢	
C. Berry	8	8	7	8	8	3	42	60¢	
H. Ray	4	8	8	6	8	4	38	25¢	
Total									

The amount that each earned depended upon the wage per hour and .

The Statistical Graph. — In business, in school, and in many other places facts about attendance, pupil's marks,

Tabulating Statistics

wages, and other things must be recorded and tabulated. These become much clearer when pictured, because the *graph* shows vividly the relations between them.

PART I. THROUGH SIMPLE EQUATIONS

CHAPTER I

FORMULAS

THIS picture shows the new bridge across the Hudson River in New York, one of the greatest achievements of modern engineering. It will cost over $50,000,000, will be

Courtesy Port of New York Authority

THE NEW HUDSON RIVER BRIDGE

3500 feet between the towers and 4800 feet long, and its towers will be 650 feet high. The weight of the suspended structure, 200 feet above the water, will be 90,000 tons, carried on cables 3 feet in diameter.

A few years ago the men who planned this bridge were in school studying algebra, as you are now. Without a knowledge of algebra, and the ability to use formulas, to read tables, and to construct graphs, they could not build this bridge. The architect who plans and the engineer and the contractor who build bridges, subways, tunnels, or great buildings, must know algebra. So also must the scientist, the surveyor, and the navigator, and a knowledge of it is very useful to the banker, the broker, and all who engage in "Big Business."

Algebra is the *bridge* which leads from arithmetic into the realms of higher mathematics. Algebra is a type of shorthand to help make your problems easier. You will use algebraic shorthand just as the stenographer uses certain symbols to stand for words or groups of words.

Perhaps you have a shorthand of your own. One boy wrote that he attended L. J. H. S. but his brother was at M. I. T., meaning Lincoln Junior High School and Massachusetts Institute of Technology. Do you blame him for using shorthand for these names?

1. Algebra as Shorthand. — Like the shorthand of this boy algebra often uses initial letters to stand for numbers; for example, l for the number of units of length, w for the number of units of width, A for the number of units of area, n for any number, etc. In using letters as well as the Arabic figures of arithmetic, algebra simplifies the solution of problems and offers many short-cuts in working examples.

2. Symbols. — Algebra also uses certain symbols to stand for words denoting operations or relations. Most of these you have already had in arithmetic. The following is a list of some of them.

Symbol	Meaning
$+$	Plus, more than, added to, increased by, the sum of.
$-$	Minus, take away, subtracted from, diminished by, the difference of.
$2 \times a$, $2 \cdot a$, $2\,a$.	Two times a, a multiplied by 2, the product of a and 2.
$\frac{m}{n}$, $m \div n$, $m : n$.	m divided by n, the quotient of m divided by n.
x^2	x times x, the square of x, x square.
$=$	Equals, is equal to.

One important difference between algebra and arithmetic should be noted here. In arithmetic 23 means 2 tens *added* to 3 units. In algebra 2 a means 2 *multiplied* by a.

$x + y$ means x plus y. prt means p times r times t.
lw means l times w. $2\,\pi r$ means 2 times π times r.
πr^2 means π times r times r.
$5(a + b)$ means 5 times the quantity $(a + b)$.
$6\sqrt{x}$ means 6 times the square root of x.

HISTORICAL NOTE

The use of symbols to represent unknown quantities and operations began with Diophantus, who lived at Alexandria, in the fourth century, A.D. Previously all words had been written out in full. The symbols used by Diophantus were a kind of shorthand but they were not at all like those used to-day. However, they were not improved upon until the beginning of the seventeenth century, when our modern symbols began to come into use.

Exercises

Using the symbols just given, write each of the following:

1. 6 times x.
2. a times b.
3. 5 added to x.
4. The sum of m and n.

5. x minus y.
6. m decreased by n.
7. n decreased by m.
8. Product of a and x.
9. 9 times x times y.
10. a times b times c.
11. 2 divided by a.
12. a divided by 2.
13. A equals l times w.
14. P equals 2 w plus 2 l.
15. c equals n times 7 c.
16. The sum of x and y divided by 2.
17. The product of a and b taken from the product of c and d.
18. The quotient of m divided by n added to the quotient of p divided by q.
19. Five times a equals three times b.
20. x less y equals 6 times n.

Use initial letters and write in algebraic shorthand:

21. Length times width.
22. Base times height.
23. Twice the length plus twice the width.
24. Eight times a number.
25. A number increased by 5.
26. Twice a number equals 10.

Read (interpret or tell the meaning of) the following:

27. $6 + 7$.
28. $7 + x$.
29. $x + y$.
30. $9 - 7$.
31. lwh.
32. $2w + 2l$.
33. $7(a + b)$.
34. $5\sqrt{x+y}$.
35. $a - b$.
36. $2ac$.
37. $\frac{x}{2}$.
38. $x + 2y$.
39. $\frac{m+n}{2}$.
40. $a = 2c$.
41. $m + n = ab$.
42. $\frac{m}{n} = \frac{x}{y}$.
43. $2n = 12$.

3. Shorthand in the Formula. — One of the commonest uses of algebraic shorthand is the formula.

A formula is a rule expressed in symbols.

It is a shorthand way of stating relations between numbers. For instance, in arithmetic we learned that the interest on a sum of money is the product of the principal, the rate, and the time in years. Using the shorthand of algebra we may write this as a formula:

$$I = prt.$$

Similarly the area of a rectangle whose length is l, and whose width is w, may be expressed by the formula

$$A = lw.$$

This should be read "The area of a rectangle equals the length times the width." (Of course, as in arithmetic, only like units can be multiplied together, that is, we cannot multiply the length in feet by the width in inches.)

One of the advantages of algebra is its power to express briefly such relations as these. We shall now practice making simple formulas.

Oral Exercises

Make up formulas (or shorthand statements) for each of the following:

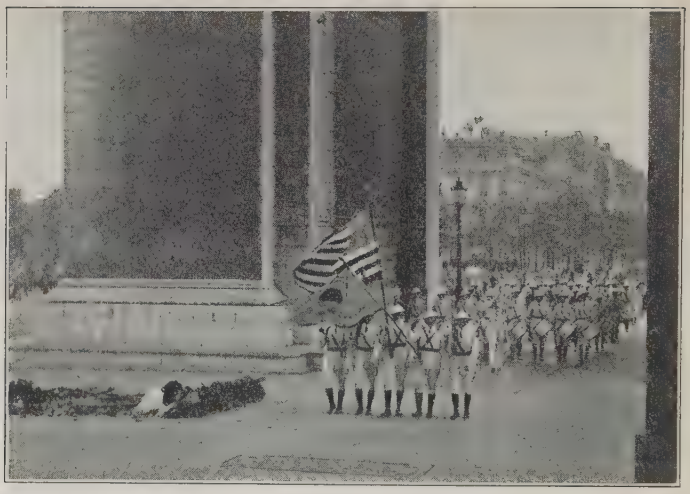
1. The number (n) of games a baseball team won if it played x games and lost 4.

2. Find the height (h) of a pile of 20 books if the thickness of each book is t inches.

3. Find the length (l) of a city block if the sidewalk consists of 50 cement squares, and the length of each square is represented by s.

4. The cost (C) of 5 notebooks at p cents each.

5. The age (A) of John 10 years hence if he is x years old now.



THE UNITED STATES FLAG

American soldiers marching past the tomb of the Unknown Soldier beneath the Arch of Triumph in Paris.

6. Find the width (w) of the United States flag if the width of each stripe is represented by s.

7. The number of cents (x) George has if he had 10 cents and spent n cents.

8. The length (l) of three rulers placed end to end if each of the rulers is i inches long.

9. The length of three rulers whose lengths are a, b, and c inches respectively.

10. A formula for coffee for (p) persons if a tablespoon (t) is used for each person and one tablespoon is added for the pot.

4. Writing Formulas from Verbal Statements. — A verbal statement of a relation between numbers can be expressed as a formula if we use a suitable letter to represent each of the numbers. Usually initial letters are best for this purpose.

For example, if the length (l) of a table is twice its width (w), we may express this as a formula: $l = 2\,w$.

Written Exercises

Using initial letters, or letters suggested in the problem, write formulas for each of the following verbal statements:

1. The circumference of a circle is equal to the diameter multiplied by π.

2. The velocity (v) of a body falling from rest is equal to 32 times the time (t) it has been falling, in seconds.

3. The rate I travel is the distance divided by the time.

4. The cost of a number of articles is their number (n) times the price (p) of each.

5. The perimeter of a regular polygon equals the length of a side (s) multiplied by the number of sides (n).

6. The volume (v) of a rectangular solid equals the length times the width times the height.

7. The diagonal of a square equals the length of a side times 1.414.

8. The yearly rent R if the rent per month is r.

9. The weight (W) of 10 packages each weighing (p) pounds.

10. The weight (W) of (n) cubic feet of pure water if one cubic foot weighs $62\frac{1}{2}$ pounds.

11. The weight in ounces (O) of (p) pounds of lead.

12. The area (A) of a circle, if its area equals the square of its radius times π.

13. The area of any triangle is half the base times the height.

14. The average (A) of any two numbers n and m is their sum divided by 2.

15. The number of months in any number of years is that number times 12.

16. The number of cents in any number of dollars is that number times 100.

17. The number of rods in any number of miles is that number times 320.

18. The number of acres in any rectangular field is the number of rods in length times the number of rods in width divided by 160.

19. The number of pounds (avoirdupois) in any number of ounces is that number divided by 16.

20. The length of the diameter of any circle is its circumference divided by π.

21. The number of 2-cent stamps I can buy for any number of dimes is 10 times that number divided by 2.

22. The number of square inches (s) in the surface of any cube is 6 times the square of the number of inches in one edge (e).

5. Evaluating (Finding the Value of) a Formula. — One great advantage of the formula is that we can find the value of any letter if we know the values of the others. This is called *evaluating* the formula for a particular letter. To do this we substitute the known values in the formula, just as in football a substitute is put in place of an injured player. This process is called *substitution*.

Illustrative example:

Evaluate the formula $A = lw$ for A, if $l = 25$ and $w = 10$.

Solution: $A = lw \qquad l = 25. \qquad w = 10.$

Substitute $= 25 \cdot 10$

$= 250.$

Rule. — 1. *Write the formula.*
2. *Write the known values at the right.*
3. *Put the numbers in the place of the letters.*
4. *Perform the indicated operations.*

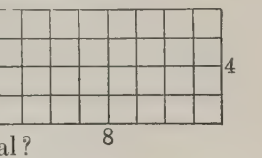
In the adjoining rectangle, answer the following questions:

If $l = 8$ and $w = 4$, what does A equal?

If $l = 16$ and $w = 4$, what does A equal?

If $l = 16$ and $w = 8$, what does A equal?

If $l = 8$ and $w = 2$, what does A equal?



If you double l when w remains the same, how is the area affected?

If you double both l and w, how is the area affected?

If you halve w when l remains the same, how does the area change? If w remains the same, the area depends upon . . .

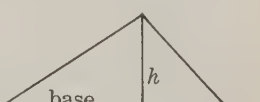
Exercises with Geometry Formulas

1. Using the formula $A = lw$, find the value of A by substitution when

(a) $l = 50$, $w = 10$.
(b) $l = 50$, $w = 20$.
(c) $l = 25$, $w = 20$.
(d) $l = 12.5$, $w = 2.5$.
(e) $l = \frac{1}{2}$, $w = \frac{1}{4}$.
(f) $l = 4\frac{1}{2}$, $w = 3\frac{1}{4}$.

2. The area of a triangle is found by taking one half the product of its base and its height. The formula is

$$A = \tfrac{1}{2}bh.$$



Using substitution in this formula, find the value of A in each of the following:

(a) $b = 12$, $h = 7$.
(b) $b = 24$, $h = 7$.
(c) $b = 36$, $h = 7$.
(d) $b = 12$, $h = 14$.
(e) $b = 24$, $h = 14$.
(f) $b = 36$, $h = 21$.

How does the area change

(g) If h remains the same and b is doubled?
(h) If h remains the same and b is trebled?
(i) If b remains the same and h is doubled?
(j) If both b and h are doubled?
(k) If both b and h are trebled?

3. Evaluate A in the formula $A = \frac{1}{2}bh$ if

(a) $b = 6.5$, $h = 3.3$.
(b) $b = 6.5$, $h = 6.6$.
(c) $b = 52$, $h = 13.2$.
(d) $b = 13$, $h = 3.3$.
(e) $b = 5\frac{1}{2}$, $h = 3\frac{1}{4}$.
(f) $b = 27\frac{1}{2}$, $h = 16.25$.

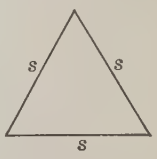
4. The formula for the perimeter of (distance around) a triangle is $P = a + b + c$.

Using this formula, evaluate P in the following:

	a	b	c	P		a	b	c	P
1.	30	20	15		3.	3	4	5	
2.	10	12	15		4.	2.6	3.5	4.2	

5. The formula for the perimeter of an equilateral (all sides =) triangle is

$$P = 3\,s.$$

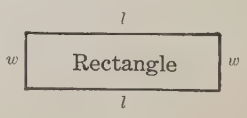


Find the value of P when the side is 5; 10; 15; 20.

How does the perimeter change if the side is doubled? Trebled? Quadrupled? Then the perimeter of an equilateral triangle depends upon . . .

6. The formula for finding the distance around a *rectangle* (called its perimeter, P) is found by adding twice its width (w) to twice its length (l).

$$P = 2\,w + 2\,l.$$



Evaluate the above formula for P when

(a) $l = 18, w = 8.$ (d) $l = 18, w = 4.$
(b) $l = 36, w = 8.$ (e) $l = 36, w = 16.$
(c) $l = 54, w = 8.$ (f) $l = 54, w = 24.$

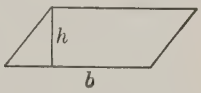
State how the perimeter changes when

(g) Both l and w are doubled.
(h) Both l and w are trebled.

The perimeter of a rectangle depends upon the . . . and the . . .

The formula for the area of a *parallelogram* is

$$A = bh.$$



Find the area of the following parallelograms:

7. (a) $b = 10, h = 6.$ (c) $b = 10, h = 12.$
(b) $b = 20, h = 6.$ (d) $b = 20, h = 12.$

How do the areas change when

(e) b is doubled and h remains the same?
(f) b remains the same and h is doubled?
(g) Both b and h are doubled?

8. (a) $b = 20, h = 2\frac{1}{2}.$ (c) $b = 60, h = 7\frac{1}{2}.$
(b) $b = 40, h = 5.$ (d) $b = 100, h = 12\frac{1}{2}.$

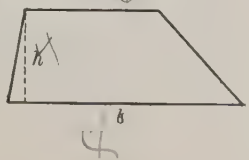
Compare results as in example 7.

9. In a parallelogram

(a) If both b and h are doubled, the area . . .
(b) If b remains the same and h is doubled, the area . . .
(c) If b is doubled and h remains the same, the area . . .

10. The formula for the area of a *trapezoid* is

$$A = \frac{h}{2}(b + b').$$ (b' is read b prime.)



Find the area of the following trapezoids:

(a) $b = 20,\ b' = 10,\ h = 4.$ (c) $b = 20,\ b' = 5,\ h = 4.$

(b) $b = 20,\ b' = 10,\ h = 8.$

Exercises with Business Formulas

1. The *cost* (C) of any number of similar articles is found by multiplying the *price* (p) of each by the *number* (n) of articles. The formula for this is

$$C = np.$$

Using this formula, find the cost when

(a) $n = 16,\ p = 8$ cents. (d) $n = 16,\ p = 16$ cents.

(b) $n = 32,\ p = 8$ cents. (e) $n = 32,\ p = 16$ cents.

(c) $n = 48,\ p = 8$ cents. (f) $n = 48,\ p = 24$ cents.

State how the cost changes when

(g) n is doubled and p remains the same.

(h) n is trebled and p remains the same.

(i) n remains the same and p is doubled.

(j) Both n and p are doubled.

(k) Both n and p are trebled.

2. The selling price (*s. p.*) is considered as the cost (c) added to the gain (g).

Write this as a formula.

Evaluate this formula for *s. p.* when

(a) $c = 15¢,\ g = 3¢.$ (d) $c = 15¢,\ g = 5¢.$

(b) $c = 12¢,\ g = 3¢.$ (e) $c = 15¢,\ g = 6¢.$

(c) $c = 9¢,\ g = 3¢.$ (f) $c = 15¢,\ g = 10¢.$

State how the selling price changes if

(g) g remains the same and c is decreased.

(h) g is increased and c remains the same.

Write a formula involving *s. p.*, c, and loss l and discuss as above.

3. The net price (*n. p.*) is the gross price (*g. p.*) less the discount (*d*). Write this as a formula.

Evaluate for *n. p.* when

(a) *g. p.* = 90¢, *d* = 9¢.
(b) *g. p.* = 80¢, *d* = 9¢.
(c) *g. p.* = 90¢, *d* = 7¢.
(d) *g. p.* = 90¢, *d* = 10¢.

State how the net price changes when

(e) *g. p.* is decreased and *d* remains the same.
(f) *g. p.* remains the same and *d* is decreased.

4. The cost (*c*) plus the gain (*g*) equals the asking price (*a*) minus the discount (*d*). Write this as a formula.

5. The formula for *simple interest* on a sum of money is the product of the principal (*p*), the rate (*r*), and the time (*t*) in years. That is, $I = prt$.

Illustrative example:

Find the interest on \$300 at 4% for 2 years.

Solution: $I = prt$ $\quad p = 300$
$= 300 \cdot .04 \cdot 2$ $\quad r = .04$
$= \$24.$ $\quad t = 2$

6. Use the above formula to find the interest on \$300 at 4% for 4 years.

How does your answer compare with that of example 5?

How does this problem differ from example 5?

Then if the principal and rate remain the same, the interest depends upon . . .

7. Use the interest formula in the following:

(a) $p = \$650, r = .04, t = 12.$
(b) $p = \$1300, r = .04, t = 12.$
(c) $p = \$650, r = .04, t = 6.$
(d) $p = \$650, r = .08, t = 24.$

Compare answers of (b), (c), and (d) with (a).

8. Copy the table below and put a t or an f after each statement that you believe is true or false. Check by solving each.

	Principal	Rate	Time in Years	Interest	Check Mark
	P	r	t	I	
a.	\$10000	.04	3	\$1200	
b.	8000	.06	$2\frac{1}{2}$	1200	
c.	900	.05	4	1800	
d.	1200	.03	5	1800	
e.	1800	$.04\frac{1}{2}$	2	162	

Problems with Distance, Time and Rate Formulas

The formula for finding the distance traveled if the rate and time are given is $d = rt$.

Illustrative example:

How far will an automobile go in 6 hours at an average rate of 30 miles per hour?

Solution: Substituting these values in the above formula, we have

$$\begin{aligned} d &= rt \\ &= 30 \cdot 6 \\ &= 180 \text{ miles.} \end{aligned}$$

Using the above formula,

1. Find d when

(a) $r = 40, t = 10$.
(b) $r = 40, t = 5$.
(c) $r = 20, t = 10$.
(d) $r = 80, t = 20$.

State how d changes when

(e) r remains the same and t is halved.
(f) r is halved and t remains the same.
(g) r is doubled and t is doubled.

2. Copy and complete the following table:

	Rate	Time	Distance
	r	t	d
a.	30	4	
b.	60	4	
c.	25	7	
d.	25	3.5	
e.	50	3.5	

Answer questions similar to those in example 1.

3. The formula for the rate is

$$r = \frac{d}{t}. \quad (d \text{ divided by } t)$$

Copy and complete the following table:

	Distance	Time	Rate
	d	t	r
a.	100 mi.	4 hr.	
b.	100 mi.	2 hr.	
c.	200 mi.	4 hr.	
d.	300 mi.	12 hr.	
e.	168 mi.	3.5 hr.	
f.	180 mi.	4.5 hr.	
g.	230 mi.	5.2 hr.	
h.	247 mi.	6.5 hr.	

State how r changes when

(i) d remains the same and t is halved.

(j) d is doubled and t remains the same.

(k) Both d and t are trebled.

4. The formula for t is $t = \frac{d}{r}$.

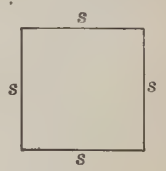
Copy and complete the following table:

	Distance	Rate	Time
	d	r	t
a.	64	$2\frac{1}{2}$	
b.	64	5	
c.	64	10	
d.	32	$2\frac{1}{2}$	
e.	16	$2\frac{1}{2}$	
f.	128	5	

Make up and answer questions similar to the above, showing how t changes under certain conditions.

Exercises with Miscellaneous Formulas

1. The square in the diagram has a perimeter of $4\,s$. The formula is $P = 4\,s$.



Find the perimeter of a square whose side is 4 inches; 8 inches; 16 inches; $10\frac{1}{2}$ miles; $2\frac{1}{2}$ inches; $6\frac{1}{4}$ centimeters.

If s increases then P . . .

2. The formula for the area of a square is

$$A = s^2. \quad (s^2 \text{ means } s \text{ times } s.)$$

Find the area of a square whose side is 5 feet; 10 feet; 20 feet; 9 inches; 12 meters.

Does A also depend upon s? How?

3. A square is 12 rods on a side. How many rods of fence will surround it? How many rods of wire will be needed if the fence has 3 strands of barbed wire? If there are 5 strands? n strands?

Interpret the meaning of the formula $w = 4\,sn$ in connection with this problem.

4. A room is 15 feet square. How many square yards of carpet will cover it? If a room is a feet square, what does the formula $y = \frac{a^2}{9}$ mean?

5. A rectangular room is 22 feet long and 14 feet wide. Write the formulas that you would use to find its area and its perimeter. Find them.

6. A rectangular room is a feet long and b feet wide.
What does ab represent?
In the center of that room there is a rug c feet square.
What does c^2 represent?
What does $ab - c^2$ represent?

7. How many square feet of the floor in example 6 are bare? How many square yards?

8. The area of this circle is expressed by the formula

$$A = \pi r^2.$$

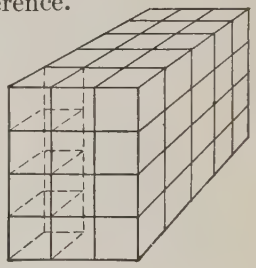
What number does π represent?
Is this value exact or only approximate?
What is its value to the nearest tenth? To the nearest hundredth?

9. $C = 2\,\pi r$ is the formula for the circumference, or the length of the circle. Find the circumference when

(a) $r = 7$ inches. (c) $r = 21$ inches.
(b) $r = 14$ inches. (d) $r = 49$ inches.

How does the circumference change when the radius is doubled? Trebled? Find the area of the above circles.
How does the area change under similar conditions?

10. Draw circles by use of several circular objects, as a coin, ink bottle, drinking glass, button, etc. Measure the radius of each as carefully as possible and by the use of the above formulas compute the circumference and area of each. Do you think your results are exact? Why? Let several pupils measure the same circular objects and see if you can agree as to the nearest tenth of a square inch as well as the exact number of inches in the circumference.



11. Find the area of a circle when

(a) $r = 3.5$. (c) $r = 13.6$

(b) $r = 4.2$. (d) $r = 2\frac{1}{3}$.

Using the formula for the volume of a rectangular solid

$$V = lwh, \quad l = \frac{V}{wh}, \quad w = \frac{V}{lh}.$$

12. Find V if $l = 40$, $w = 10$, $h = 6$.

13. Find l if $V = 3920$, $w = 14$, $h = 8$.

14. Find w if $l = 25$, $h = 5$, and $V = 2250$.

15. Find V if $l = 20$, $w = 18$, and $h = 10$.

16. Find V if $l = 20$, $w = 18$, and $h = 20$.

17. State how the volume of the rectangular solid is affected if the length and the width remain the same and the height is doubled.

Using the formula for the curved surface of a cylinder

$$s = 2\pi rh,$$

Area of curved surface.

18. Find S if $\pi = \frac{22}{7}$, $r = 7$, $h = 10$.

19. Find S if $\pi = \frac{22}{7}$, $r = 14$, $h = 10$.

20. Find S if $\pi = \frac{22}{7}$, $r = 14$, $h = 20$.

21. How does the surface of a cylinder change if the radius is doubled and the height doubled?

Using the formula for the volume of a pyramid

$$V = \tfrac{1}{3}Bh, \quad B = \frac{3V}{h}, \quad h = \frac{3V}{B}.$$

22. Find V if $B = 24$ and $h = 8$.

23. Find B if $V = 108$ and $h = 27$.

24. Find h if $V = 144$ and $B = 36$.

25. Find h if $V = 72$ and $B = 36$.

26. State how h changes when B remains the same and V decreases.

Using the formula for the volume of a prism

$$V = Bh, \quad B = \frac{V}{h}, \quad h = \frac{V}{B}.$$

27. Find h if $V = 300$ and $B = 30$.

28. Find B if $V = 600$ and $h = 30$.

29. Find B if $V = 600$ and $h = 15$.

30. How does B change if V remains the same and h decreases?

Practice Test

Evaluate each of the following formulas for the letter at the left of the = sign:

1. $A = \frac{1}{2}bh$ if $b = 16$ and $h = 2.5$.
2. $V = lwh$ if $l = 40$, $w = 15$ and $h = 12$.
3. $I = prt$ if $p = \$1250$, $r = .04$ and $t = 3\frac{1}{2}$ years.
4. $A = s^2$ if $s = 3.5$.
5. $A = lw$ if $l = 4.56$ and $w = 3.84$.

6. $C = 2\pi r$ if $r = 14$ and $\pi = \frac{22}{7}$.

7. $A = \frac{a}{2}(b + c)$ if $b = 14$, $c = 12$, and $a = 9$.

8. $s = c + g$ if $c = \$25.75$ and $g = \$5.15$.

9. $A = \pi r^2$ if $\pi = \frac{22}{7}$ and $r = 7$.

10. $K = \frac{m}{.6213}$ if $m = 26.21629^{+}$. (Marathon run.)

6. Interpretation of Formulas. — Now that we have had practice in evaluating formulas and in making them from verbal statements, we should be able to understand the meaning of a formula when we see it, and to express that meaning in words. For example, when we see the formula $A = \frac{1}{2}ba$ for the area of a triangle, we should say "This means that the area of a triangle is equal to half the product of the base and height."

Exercises

State in words the meaning of each of the following formulas.

1. For the square, $p = 4s$, $A = s^2$.
2. For the rectangle, $p = 2l + 2w$, $A = lw$.
3. For the trapezoid, $A = \frac{1}{2}h(b + b')$.
4. For the circle, $A = \pi r^2$, $c = 2\pi r$, $A = \frac{1}{4}\pi d^2$, $c = \pi d$.
5. For the cube, $V = e^3$ (e = edge); S (surface) $= 6e^2$.
6. For the sphere, $S = 4\pi r^2$, $V = \frac{1}{6}\pi d^3$, $V = \frac{4}{3}\pi r^3$.
7. For the rectangular solid, $V = lwh$.
8. For the pyramid, $V = \frac{1}{3}bh$.
9. For the prism, $V = bh$.
10. For the cylinder, $V = \pi r^2 h$.
11. For the cone, $V = \frac{1}{3}\pi r^2 h$.

12. For falling bodies, V (velocity) $= 32\,t$ (time in seconds).

13. For interest, $I = prt$, $A = p + prt$.

14. For finding distance, $d = rt$.

15. In business, $g = s.\ p. - c$, $l = c - s.\ p.$

7. Formulas Showing Dependence. Function. — In a formula the value of one quantity depends upon the value of some other quantity. If we buy eggs at 30 cents a dozen, their cost in cents is represented by the formula

$$c = 30\,d.$$

Here the cost depends on the number of dozen bought, that is to say, *c depends upon d.*

In the same way, if we buy 6 dozen eggs, the cost depends on the price per dozen. The formula for this is

$$c = 6\,p,$$

and *c depends upon p.*

How is c affected if p is increased? If p is decreased?

When one quantity depends for its value on another quantity, the first is called a *function* of the second.

Exercises in Functions

1. If d equals the distance I traveled at 30 miles an hour and t represents the number of hours, then $d = 30\,t$. By means of this formula we may solve such a problem as "How far can a man travel in 8 hours going 30 miles an hour?"

Write two more problems which can be solved by this formula.

2. $c = 19\,g$. c = the cost of gasoline at 19 cents a gallon and g = the number of gallons.

Write three problems that can be solved by this formula.

3. $d = \frac{100}{p}$. d = the number of dozen oranges I can buy for a dollar, p = the price per dozen.

Write three problems that can be solved by this formula.

4. $g = \frac{m}{15}$. g = the number of gallons of gasoline necessary to drive my car m miles, if I get 15 miles to the gallon.

Write three problems of which this formula is a general solution.

5. Upon what does d depend in example 1? How is d affected if t is increased? if t is decreased?

6. Upon what does c depend in example 2? How is c affected if g is increased? if g is decreased?

7. Upon what does d depend in example 3? How is d affected if p is increased? if p is decreased?

8. In example 4 of what is g a function? How is g affected if m is increased? if m is decreased?

9. $c = \frac{22}{7} r$. Of what is c a function? How is c affected if r is increased? if r is decreased?

10. $A = \frac{1}{2} bh$. Upon what two things does A depend?

Review

1. Express d dollars, q quarters, and c cents as cents.

2. Make a dual-bar graph to represent the number of ships passing through the Panama canal and the tolls in 1927, also in 1928.

Year	Number of Ships	Tolls
1927	6085	$26,231,022
1928	6334	$26,375,692

THE PANAMA CANAL

Write formulas for

3. The dividend D if the divisor d, the quotient q, and the remainder R are given.

Complete the following:

4. The perimeter of a square depends upon . . .

5. The amount of heat that you receive from a stove having a fixed amount of heat in it depends upon . . .

6. The selling price of a small article should depend upon . . . If the number sold increases, then the price should . . .

7. The amount of light that you get from a certain lamp depends upon . . .

CHAPTER II

FORMULAS AND EQUATIONS

In the preceding chapter we learned of algebra as shorthand, and saw that the formula is one of its most useful applications. We learned how to evaluate formulas and algebraic expressions and so prepared ourselves to go out into the business world and compute the volumes of solids, areas of surfaces, the amount of interest, and many other practical applications of formulas.

We shall now learn how scientists, economists, and others are able to obtain formulas from the study of information arranged in tables, how formulas are graphed and from that graph how one quantity in a formula depends upon another. All this will lead up to a study of the most important subject in algebra — the equation. Of the equation we shall take up in this chapter only the simplest types as preparation for our further study.

8. Dependence of Related Numbers. — When a number is dependent upon a second number, the two are definitely related. We have learned how the area of a rectangle changes when the base remains the same and the height increases or decreases. We have also seen how the perimeter of a square changes as the side increases in value.

These related numbers may be illustrated by tables such as we shall now study and complete. This will show us how one number changes when there is a change in another number on which the first depends. From a study of these changes we can write a formula from the table.

Illustrative example:

George works in a garage where the following table can be used.

Gallons	1	2	3	4	5	6	7	8	9	10
Cost .	20¢	40¢	60¢	80¢	$1.00	$1.20	$1.40	$1.60	$1.80	$2.00

Solution: The numbers in each column of the table are related as 1 to 20. If we let n equal the number of gallons purchased, and study the related numbers in the table, we can write a formula for the cost: $C = \$.20\, n$.

Exercises

The first 5 questions refer to the above table or a similar one.

1. If I buy 6 gallons, where will George look for the cost? How much is it?

2. How would you find the number of gallons I can buy for $1.60?

3. How would you find the cost of 20 gallons?

4. How does the cost change as I buy more and more gallons?

5. Write a formula for the cost of gasoline at 18 cents a gallon.

6. John receives $12 a week. Copy and complete the table:

Weeks	1	2	3	4	5	6	7	?	9	?
Wages	$12	$24			$60	$72		$96	$108	$120

How do the wages change as the number of weeks increase? Then the total wages will depend upon the number of

7. Write a wage formula for John.

8. Henry deposited 25 cents a week regularly in his School Savings account. Copy and complete the table:

Weeks	1	2	3	4	5	6	7	8	9	52
Savings	25¢	50¢	75¢	$1.00						

Savings increase as the number of weeks . . .

How much will he have saved at the end of 26 weeks? Of 52?

Write a formula showing his savings for w weeks.

9. Copy and complete the table:

Pounds of Sugar . . .	1	2			5	
Cost . . .	6¢		18¢	24¢		36¢

As the number of pounds increase the cost . . .

Write a cost formula for sugar at 6¢ a pound. At 7¢. At 8¢.

10. A railroad train averages 40 miles per hour. Copy and complete the following table:

Hours	1	2					
Miles	40	80	120	160	200	240	280

How does the distance change when the time is increased?

Write a distance formula for the above. Also one at 50 miles an hour. At 25 miles an hour.

11. The distance traveled in miles is the result obtained by multiplying the average number of miles per hour by the number of hours. Copy and complete the following:

Rate per hour . . .	25	25	25	25	28	30	32	35
Hours	1	2	3	4	4	4	4	4
Miles								

If the rate per hour remains the same and the time increases, the distance . . .

If the time remains the same and the rate increases, then the distance traveled . . .

12. You have learned how to find the circumference of a circle. $C = 2\pi r$. Complete the following:

Radius . . .	7	14	21	28	35	42	49
Circumference .	44	88	132	176	220		

What happens to the circumference when the radius is doubled? Trebled? Quadrupled?

13. You also learned how the area of a rectangle is obtained. Copy and complete the table:

Base . .	20	20	20	20	20	20	20	20
Altitude .	2	4	6	8	16	32	64	128
Area . .	40	80	120					

If the base remains the same, how does the area change when the altitude is doubled? Trebled? Quadrupled? Multiplied by 8? By 32?

Give the formula for the area of a rectangle.

14. The area of a square is obtained by squaring a side. Copy and complete the following table:

Side . .	1	2	3			6		
Area . . .	1	4		16	25	36	49	64

If the area of a square decreases, the length of the side . . .

Try to write a formula for the area of a square.

How do you find a side from the area?

9. Obtaining Formulas from the Related Numbers of a Table. — It is often necessary, especially in the study of science, to obtain a formula from the study of a table showing how one quantity is related to another, that is, how it depends upon another.

Here is a simple illustration:

A photographer charges 15 cents for developing a roll of films and 6 cents for each print he makes. We have then the following table:

Number prints	0	1	2	3	4	5	6	7	8
Cost	15¢	21¢	27¢	33¢	39¢	45¢	51¢	57¢	63¢

In this table we see that each number in the lower row of numbers is 6 times the corresponding number in the upper row plus 15. Then if c represents the cost in cents, and n the number of prints, the formula is

$$c = 6n + 15.$$

Another illustration:

t	1	2	3	4	5
s	1	4	9	16	25

This table means that when $t = 1$, $s = 1$; when $t = 2$, $s = 4$, etc. A study of the table shows that every number in the lower line is the square of the corresponding number in the upper line. Hence we have the formula

$$s = t^2. \quad (t^2 \text{ means } t \cdot t.)$$

Exercises

By a study of the following tables showing the relation of two quantities, write a formula showing that relation. In each case consider the quantity in the lower line as depending upon that in the upper line.

1. The cost of sending a parcel post package in the first or second zone is as follows. Copy and complete:

Weight in Pounds	1	2	3	4	5	6	7	8
Cost	6¢	8¢	9¢	10¢				

As the number of pounds increase in order the cost increases ____ cent a pound.

Write a parcel post formula for the above.

2.

x	1	2	3	4	5
y	4	6	8	8	12

3.

a	0	1	2	3	4	5
b	0	3	6	9	12	15

4.

x	1	2	3	4	5
y	4	8	12	16	20

5.

x	1	2	3	4	5
y	2	3	4	5	6

6.

x	0	1	2	3	4	5	6
y	1	3	5	7	9	11	13

7.

x	1	2	3	4	5	6	7
y	2	5	8	11	14	17	20

In the following, write a formula showing the dependence of the quantity in the lowest line on both of the other quantities.

8.

x	1	2	3	6	8	3
y	2	3	5	4	2	6
z	3	5	8	10	10	9

9.

b	1	2	5	3	4	3	6
h	2	3	4	6	8	8	6
A							

10.

b	2	4	5	8	6	10	5	6
h	3	4	4	6	3	4	8	4
A	3	8	10	24	9	20	20	12

Additional work of this kind will be found on page 479.

10. Formulas as General Solutions. — When we were beginning the study of arithmetic, we sometimes found a page of problems like the following:

Find the cost of 6 oranges at 3 cents each.
Find the cost of 8 pencils at 5 cents each.
Find the cost of 12 pairs of shoe laces at 5 cents each.

In each of these problems we obtain the answer by multiplying the price of one article by the number of articles. Hence, if we denote the number of articles by n, the price of each by p, and the total cost by c, we have the statement

$$c = np.$$

This formula provides a solution for all problems of the above type. From these formulas or general statements we can form countless specific problems with definite numbers.

Illustrative example:

Write three problems with numbers which can be solved by the formula $p = 2l + 2w$.

Solution: 1. Find the perimeter of a rectangle whose length is 10 and width 6.

2. Find the perimeter of a rectangle whose length is 16 and width 4.

3. Find the perimeter of a rectangle whose length is 20 and width 14.

Exercises

Write three specific problems with numbers for which each of the following formulas is a general solution:

1. $d = rt.$
2. $C = 2\pi r.$
3. $V = bh.$
4. $A = s^2.$
5. $s.\,p. = c + g.$
6. $V = lwh.$
7. $i = prt.$
8. $v = 16\,t.$
9. $V = e^3.$
10. $A = \frac{1}{2}bh.$

11. The Formula Graph. — In the Introduction we studied statistical graphs as pictures of arithmetical facts. They showed graphically certain number relations some of which were in no way dependent upon one another. We are now going to take up a different type of graph, called *functional*, because it shows the dependence of one quantity (function) upon another. Functional graphs are algebraic, because they may be used to solve problems; statistical arithmetical graphs are often used merely to illustrate number relations. In all functional graphs we use the kind of graph paper that we used for statistical broken-line and curved-line graphs.

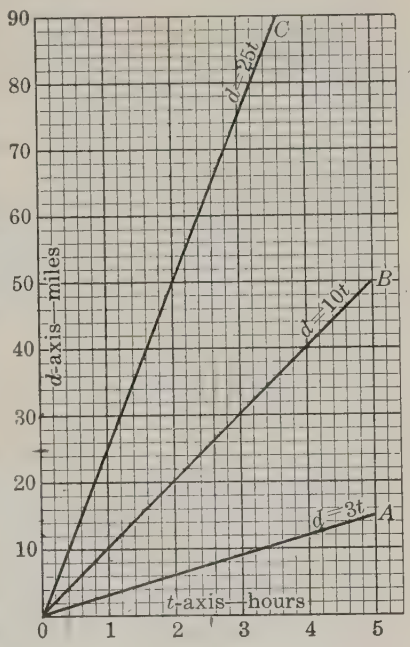
The simplest functional graph is the graph of the formula. In the formula $d = rt$, d depends on both r and t. If, however, we give r a fixed value, as 10, then our formula becomes $d = 10\,t$. In this case d depends on t only, so d is a *function* of t. We may draw a graph of this formula or of any similar one.

To construct the graph of the formula $d = 10\,t$, make a table of values of t and the corresponding values for d just as we did for Gallons and Cost or for Weeks and Wages in the section on Dependence of Related Numbers (page 25).

$t =$	1	2	3	4	5
$d =$	10	20	30	40	50

Now take a sheet of graph paper like that used in the broken-line or curved-line graphs. Call the horizontal axis the t-axis and the vertical axis the d-axis. On the t-axis mark off equal spaces to represent equal intervals of time. In this case let each large square represent 1 hour. On the vertical axis (the d-axis) mark off equal spaces to represent equal distances, letting each large square represent 10 miles and each small square 2 miles.

On the vertical line exactly above each specified hour locate a point opposite the number of miles traveled in that time. This will be 5 small spaces up from 1 on the t-axis, 10 small spaces up from 2 on the t-axis, 15 spaces up from 3, and so on. When we join these points, we get a straight line. For this reason such a formula is called a *linear formula*. Similarly, the graphs of $d = 3\,t$ and $d = 25\,t$ may be constructed. Notice that the greater the rate (for the same time) the greater the steepness or *slope* of the line. Suppose that a line were horizontal, what rate would it represent?



The advantage of these functional graphs is that they help us to solve problems. For example, let us locate two or more points from our table and connect them by a straight line. Now we can continue this line indefinitely, and if our work is accurate, it will show us *any* value of d for the corresponding value of t, or *any* value of t for the corresponding value of d.

Exercises

Read from the graphs the answers to the following questions:

1. How long will it take the man to walk 12 miles? 10 miles? 4 miles? Use graph A.

2. How long will it take him to ride his bicycle 15 miles? 25 miles? 32 miles? Graph B.

3. How long will it take him to drive his automobile 60 miles? 40 miles? 15 miles? Graph C.

4. How far can the man walk in $2\frac{1}{2}$ hours? In 3.1 hours? In $4\frac{1}{5}$ hours? Use A.

5. How far can he ride his bicycle in $1\frac{3}{4}$ hours? In $3\frac{1}{2}$ hours? In 2.7 hours? Use B.

6. How far can he drive his car in 1.7 hours? In 2.3 hours? In $3\frac{1}{3}$ hours? Use C.

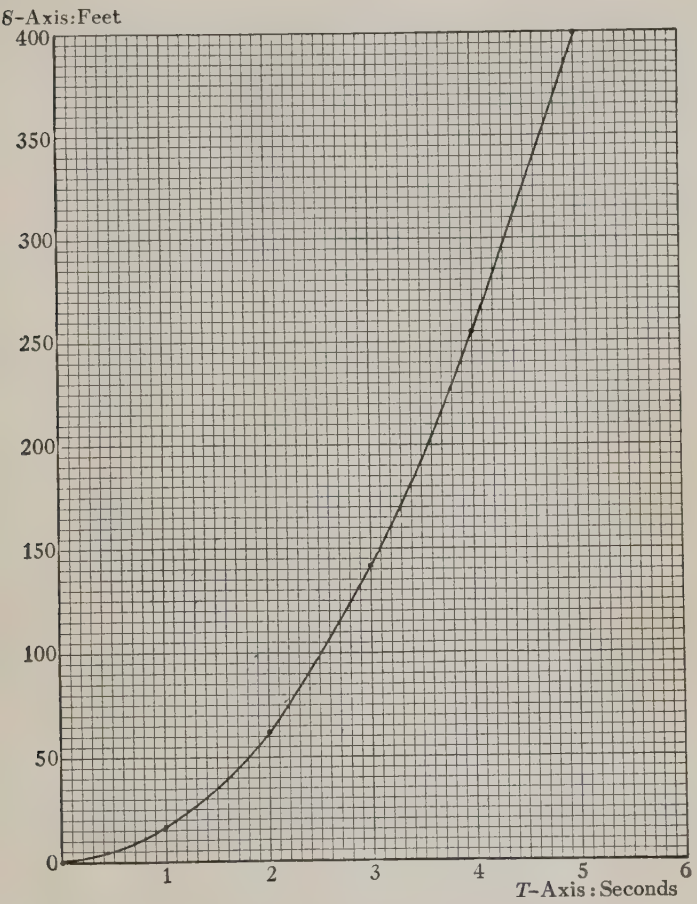
7. Construct the graphs of $d = 2\,t$, $d = 5\,t$, $d = 30\,t$, and $d = 100\,t$.

Not all formula graphs are straight lines. For example, when a body falls, it falls faster and faster the farther it falls. The relation between the distance a body falls and the time it is falling is represented by the formula $S = 16\,t^2$. S represents distance in feet, and t represents time in seconds. By substituting successive values of t in the above formula, we obtain corresponding values of S, just as on page 32 we obtained corresponding values for d and t. These values should be arranged in a tabulated form as shown in the following table.

t	0	1	2	3	4	5
S	0	16	64	144	256	400

To graph this formula let us call the horizontal axis the t-axis, and the vertical axis the S-axis. Then we must locate on our graph paper the corresponding values of S and t, just as we did in the previous graph for d and t. When we

connect these points of location, we shall have a curved line.



By studying the graph answer the following questions:

How far will a body fall in $2\frac{1}{2}$ seconds? In $4\frac{1}{2}$ seconds?

How long will it take a body to fall 40 feet? 150 feet? 300 feet?

Exercises

Make tables, then draw graphs to represent each of the following formulas, and discuss them as directed by your teacher:

1. $C = \frac{22}{7} D$, where D = diameter of a circle and C = circumference.

2. $V = 32\,t$, where V = velocity or rate of motion and t = time in seconds.

3. $w = \$.50\,h$, where w = wages and h = hours.

4. $r = np$, where r = receipts, $n = 200$, and p = price per article.

5. $c = \$.40\,d$, where c = cost and d = number of dozen of eggs.

6. The wage formula $w = .50\,N$ if paid 50 cents an hour. Use $N = 0, 2, 4, 6,$ and 8.

From your graph, read the wages earned in $2\frac{1}{2}$ hours. In 7 hours.

From the graph find the number of hours worked when the wages are \$3.50.

How is W changed when N is doubled? Trebled?

7. Make a graph of the interest I, on \$100 at 5% for n years from the formula $I = 5\,n$.

Use $n = 0, 1, 2, 3, 4, \cdots$ to $n = 8$.

What happens to I when n is doubled? Trebled? Decreased? Show this with a rectangular or circular distribution graph.

8. Make a graph of the formula $C = \$.20\,N$ from $N = 1$ to $N = 4$.

Hint: Use vertical axis for C.

From your graph find the cost of $2\frac{1}{2}$ pounds.

9. If a man travels at the rate of 20 miles an hour, then

the formula $d = 20\,t$ shows the relation between the distance (d) and the time (t).

(a) Make a table which shall show the values of d corresponding to $t = 1, 2, 3, 4, 5$.

(b) Make a line graph of the above.

(c) Show by an x the point on your graph that represents the number of hours required to go 50 miles.

Optional Exercises

10. $A = \frac{22}{7} r^2$, where r = radius of circle and A = area of circle.

11. $m = \dfrac{10}{n}$.

12. $A = s^2$, where A = area and s = side of a square.

13. $V = e^3$, where V = volume of a cube and e = edge of the cube.

14. $V = 200 - 32\,t$, where V = velocity of a body thrown upward with an initial velocity of 200 feet per second, at the end of t seconds. $t = 0, 1, 2, 3, 4, 5, 6$.

15. $V = 4(l - 8)^2$, where V = volume of a cardboard box made from a piece of cardboard l inches square, and 4-inch squares are cut out before folding the box. Use values for l greater than 8.

12. Algebraic Expressions. — All the formulas we have been studying are called *algebraic expressions*. Any quantity represented by algebraic symbols is an algebraic expression. However, algebraic expressions are not called formulas, unless they state a rule. The following are illustrations of algebraic expressions:

$$a,\ a + b,\ x - y + z,\ 2\,abc,\ 3\frac{x}{y},\ x - \frac{7\,m - n}{4\,x + y},$$

$$2\,x^2 = hy - z,\ x^2 + 2\,xy + y^2 = \frac{ab + c}{mn - r}.$$

13. Interpreting Algebraic Expressions. — In thinking of our formula work as algebraic shorthand we used letters and certain symbols to represent words, numbers, and tabulated forms. Then we interpreted the symbols of the formulas. In the same way we are now going to express and interpret other algebraic expressions, not formulas.

For example, $\frac{3x - 2a}{n}$ means the result of subtracting two times the number a from three times the number x and then dividing that difference by the number n.

Exercises

Express by use of algebraic symbols each of the following word expressions:

Illustrative example:

A number l less than the number x expressed algebraically is: $x - l$.

1. One half the product of x and y.
2. The sum of a and b divided by $2c$.
3. The result of subtracting x from three times the square of y.
4. The quotient obtained by dividing $3a$ by the sum of b and c.
5. The sum of the quotients obtained by dividing a and b by x and y respectively.
6. The square of a diminished by the square of b.
7. The sum of twice a, three times b, and 4 times c.
8. John's age a years ago if he is n years old now.
9. Stephen is now x years old. Write an expression which will represent his age 4 years after he is twice as old.
10. A number is n greater than the number v.

Interpret in words the meaning of the following expressions:

Illustrative example:

$2\,b^2 - c^2$ means the result of subtracting the square of c from two times the square of b.

11. $\dfrac{a-b}{x}$.

12. $3\,x + y^2$.

13. $\dfrac{1}{a} + \dfrac{1}{b}$.

14. $\dfrac{a}{b} \div \dfrac{c}{d}$.

15. $\dfrac{m+n}{m-n}$.

16. $\dfrac{x^2 - y^2}{2}$.

17. $\frac{1}{3}x + \frac{1}{2}y - \frac{1}{4}z$.

18. $\dfrac{x+y+z}{3\,n}$.

19. $\dfrac{1}{a^2 + b^2 + c^2}$.

14. Preparing to Evaluate (Find the Value of) Algebraic Expressions. — In order to evaluate algebraic expressions, we must review our former work in simplifying numerical expressions. These are simplified by the following

Rule. — *In numerical operations, the processes of multiplication and division must always be performed in the order indicated before the operations of addition and subtraction are performed. Numerators and denominators must be worked first.*

My Dear Aunt Sally	Rule	Multiplication Division Addition Subtraction	This means that all multiplications and divisions must be done before the additions and subtractions.

Illustrative example I:

$$\begin{aligned} 10 \div 2 \cdot 8 - 4 \cdot 10 \div 2 &= 5 \cdot 8 - 40 \div 2 \\ &= 40 - 20 \\ &= 20. \end{aligned}$$

Illustrative example II:

$$\begin{aligned} 5 \cdot 6 - 12 \div 2 + 8 \div 2 \cdot 5 &= 30 - 6 + 4 \cdot 5 \\ &= 30 - 6 + 20 \\ &= 44. \end{aligned}$$

Oral Exercises

Find the value of:

1. $4 \cdot 3 + 2$.
2. $3 \cdot 6 - 4$.
3. $2 + 3 \cdot 6 - 4$.
4. $15 - 2 \cdot 7 + 6$.
5. 6 times 1 ft. + 8 times 1 ft.

Exercises

In accordance with the rule above find the value of each of the following expressions:

1. $5 + 5 \cdot 5 - 5 \div 5 + 5 \div 5 \cdot 5$.
2. $4 \cdot 8 \cdot 3 \div 16 + 10 \cdot 5 - 8 \cdot 6$.
3. $16 \cdot 4 - 3 \cdot 2 \cdot 5 \div 15 + 5 \cdot 3$.
4. $\dfrac{8 \cdot 6 - 10}{2} - \dfrac{4 + 3 \cdot 6}{11}$.
5. $\dfrac{30 - 5 \cdot 3}{10} + \dfrac{2 \cdot 8 - 4}{8}$.
6. $8 \cdot 10 \div 2 \cdot 3 - \dfrac{6 \div 2 \cdot 100}{5}$.

15. Evaluating (Finding the Value of) Algebraic Expressions. — We may find the numerical value of literal expressions, when the value of each of the letters is known. As in the case of the formula, the process of putting the numerical value in place of a letter is called *substitution*.

If $a = 3$ and $b = 5$, what does $a + b = ?$ $b - a = ?$ $2a + b = ?$ $\dfrac{5a}{b} = ?$ $5a - 2b = ?$ $3b - 5a = ?$

$5a$ times $3b = ?$ $6a$ times $2b$ and the result divided by $5a$ = what?

Rule. — *In place of each literal quantity substitute its numerical value and simplify the result.*

Illustrative example I:

Find the value of $3a + 4b$ when $a = 2$, $b = 10$.

Solution:
$$\begin{aligned} 3a + 4b &= 3 \cdot 2 + 4 \cdot 10 \\ &= 6 + 40 \\ &= 46. \end{aligned}$$

Illustrative example II:

Find the value of $\frac{6a}{2} - \frac{b + 3xy}{a}$ when $a = 4$, $b = 10$, $x = 2$, and $y = 6$.

Solution:
$$\begin{aligned} \frac{6a}{2} - \frac{b + 3xy}{a} &= \frac{6 \cdot 4}{2} - \frac{10 + 3 \cdot 2 \cdot 6}{4} \\ &= 12 - \frac{10 + 36}{4} \\ &= 12 - \frac{46}{4} \\ &= 12 - 11\tfrac{1}{2} \\ &= \tfrac{1}{2}. \end{aligned}$$

Exercises

Using both of the rules on pages 39 and 40, find the value of each of the following literal expressions:

Given $a = 2$, $b = 3$, $c = 5$, $d = 1$, $e = 0$, and $f = 6$.

1. $f - 4$.
2. c times f.
3. a times b times f.
4. The product of b and c.
5. The quotient of f divided by b.
6. $b + 2c$.
7. $f \div a$.
8. $6b + 5a$.
9. $f - c$.
10. $3a + 4b$.
11. $c \div d$.
12. $bc + df$.
13. $\frac{3a + 4f}{c}$.
14. $7d - b$.
15. $6e + 5f$.
16. b times b added to $2f$.

17. b times b and the result subtracted from abf.

18. What number multiplied by itself equals a times b times f?

19. $\dfrac{5a + 8d}{b}$.

20. $\dfrac{abc - 3df}{ad}$.

21. $\dfrac{3}{b} + \dfrac{ace}{c}$.

22. $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

23. $\dfrac{bc}{a} - \dfrac{ab}{c}$.

24. $\dfrac{de + c}{ab - c}$.

25. $\dfrac{def}{a} + \dfrac{baf}{3}$.

26. $\dfrac{abc}{df} - \dfrac{df}{b} + \dfrac{deb}{c}$.

27. $\dfrac{cf}{a} \cdot \dfrac{3db}{c} \div \dfrac{f}{a}$.

28. $\dfrac{4ac}{bd} - \dfrac{a}{d} \cdot \dfrac{3b}{f}$.

29. $\dfrac{3a}{c} + \dfrac{3b}{d} + \dfrac{3c}{f}$.

30. $\dfrac{ab + bc + cd + de + ef}{a + b + c}$.

31. $\dfrac{3ad}{f} - \dfrac{f - cd}{d}$.

32. $2adf - \dfrac{10f}{bc} + \dfrac{abc}{5d} \div \dfrac{d}{f}$.

Matching Exercise

Copy the numbers of the examples in the longhand group and at the right of each write the letter of the shorthand group that you know fits the corresponding number of the first group:

Longhand	Algebraic Shorthand
1. Four times the number n.	**a.** $\frac{1}{2}n$.
2. The number n diminished by 7.	**b.** $\dfrac{n}{4}$.
3. One half of the number n.	**c.** $\frac{1}{4}n = 6$.
4. The number n divided by 4.	**d.** $8 + 3n$.
5. The quotient of x divided by y.	**e.** $4n$.
6. 8 added to three times the number n.	**f.** $n - 7$.

Longhand	Algebraic Shorthand
7. Twice the number n and result added to 3.	g. $n + 9$.
8. One third of the number n diminished by 8.	h. $\frac{x}{y}$.
9. One fourth of the number n equals 6.	i. $\frac{1}{3}n - 8$.
10. The number n increased by 9.	j. $2n + 3$.

Review

1. What is the total width of 5 boards, each of which is w feet wide?

2. What is the perimeter of a rectangle whose length is x inches and whose width is y inches?

3. Write in algebraic symbols:

Five times m added to the product of $3x$ and y.

4. What is the perimeter of a square whose side is s?

5. If an automobile goes n miles an hour for t hours, how far does it go all together? What is this distance if $n = 24$ and $t = 8$?

Write formulas or algebraic shorthand for each of the following:

6. The percentage p is the product of the base B by the rate r.

7. The number of cents c in d dimes and q quarters.

8. y yards and f feet as inches i.

9. Half of the circumference of a circle with radius r.

10. The diameter of a circle is twice the radius r.

11. The total cost (C) of n articles at \$5 each.

12. The perimeter (P) of any triangle is the sum of the sides a, b, and c.

13. The perimeter of an equilateral triangle is three times a side s.

14. The cost (C) of k kilowatts of electricity at 7¢ a kilowatt.

15. The cost (C) of n pencils at p cents each.

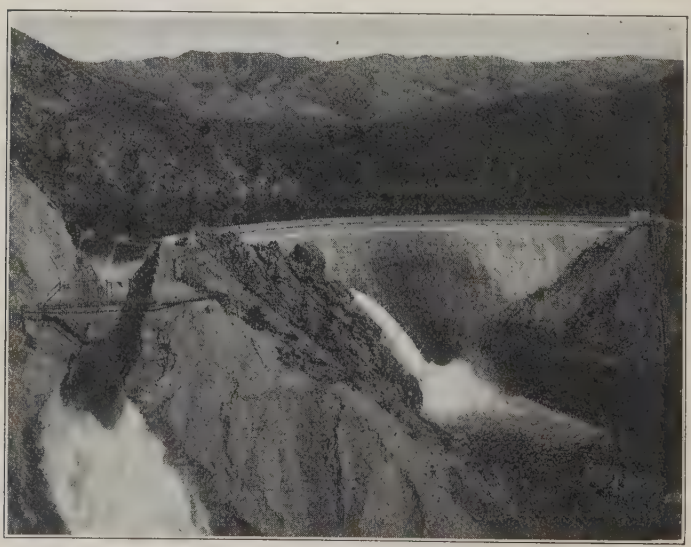
16. The rent (r) if it is one fourth of the yearly income (I).

17. Horse power ($H.\ P.$) is the product of the volts v by the amperes a and the result divided by 746.

Related Number Problems. Complete

18. When you buy sugar at a given price per pound, the cost depends upon . . .

19. The success of the Chief Engineer of the Arrow Rock Dam depended upon . . . State 3 things.



Arrow Rock Dam, Idaho

20. The perimeter of an equilateral triangle depends upon . . .

21. The time that I take to go 200 miles depends upon . . .

22. My marks in algebra will depend upon . . . State 4 items.

23. The area of a rectangle whose base is 10 inches depends upon . . .

16. Equations. — We have just studied, interpreted, and evaluated formulas and other algebraic expressions. We are now ready to take up the most important of all algebraic expressions — the equation.

An equation is a statement in symbols of the fact that two algebraic expressions are equal. For example, $A = bh$, $2x + 1 = 7$, and $a^2 = 2ab$ are all equations. Formulas are equations of one kind, and the first equation mentioned above, $A = bh$, is a formula. However, there are other equations which are not formulas, such as $2x + 1 = 7$. In this case x is not an initial letter representing some particular class of quantities; but it represents an *unknown number* whose value is to be found. To solve an equation means to find the value of the unknown quantity. So important are equations in algebra that algebra is sometimes defined as *the science of the equation*. To make the work clearer and easier this book is divided into two parts, the first leading through simple equations, the second through *quadratic* equations. Before starting this work it is best to see some illustrations of just what kinds of equations we may meet. Equations occur in four types or combinations of these types. We shall treat them briefly here, solving them by arithmetical analysis.

TYPE I. $ax = b$

Illustrative example I:

Find the value of x in $5x = 15$.

Solution: $5x = 15.$

Think, what number multiplied by 5 gives 15?

The number is 15 divided by 5, or 3.

Hence $x = 3.$ *Ans.*

Check: Put 3 in the place of x in the original equation.

$$5 \cdot 3 = 15.$$
$$15 = 15.$$

Illustrative example II:

Solve $\frac{5}{6}x = 25$ for x and check.

Solution: $\frac{5}{6}x = 25.$

Think, what number multiplied by $\frac{5}{6}$ gives 25, or 25 is $\frac{5}{6}$ of what number.

$$x = 25 \div \tfrac{5}{6} = 25 \cdot \tfrac{6}{5} = 30. \quad \textit{Ans.} \quad \textit{Check.}$$

Exercises

Interpret as a question, solve for the letter by arithmetical analysis, and check. Work mentally when you can:

1. $3x = 18.$	**4.** $12x = 72.$	**7.** $9 = \frac{3}{4}x.$
2. $10x = 40.$	**5.** $400 = 20x.$	**8.** $10\frac{1}{2}x = 21.$
3. $13x = 91.$	**6.** $\frac{1}{2}x = 4.$	**9.** $2.5x = 50.$

Problems Applying Type I

Illustrative example:

What number multiplied by 12 gives 132?

Solution: Let n = the given number.

Then $12n = 132.$

$n = 11.$ *Ans.*

Check in the problem. $11 \cdot 12 = 132.$

Solve by the equation method:

1. Five times a number equals 40. Find the number.

2. Four times the average hourly speed of a fast motor boat was 116 miles. Find the average hourly speed.

FAST MOTOR BOATS

3. $2\frac{1}{2}$ times a number equals 25. Find the number.

4. 6% (.06) interest on a sum of money is \$720. Find the sum.

5. 150% (1.5) times the cost of a toy is \$1.50. Find the cost.

6. Five sixths of a number is 50. What is the number?

Hint: Let n = number, then $\frac{5}{6}\,n = 50$.

7. The overhead expenses (\$25.36) of a firm for one day were one fourth of the receipts. Find the receipts.

8. If four fifths of the general expenses of a firm are \$5062.16, what are the general expenses?

9. Find the cost of a radio set when $\frac{3}{5}$ of the cost is \$57.

10. $\frac{1}{2}\%$ ($\frac{1}{200}$) of the cost is 6 cents. Find the cost in cents.

11. After going four fifths of my trip, I have 80 miles yet to travel. How many miles had I to travel all together?

12. 40% of the assessed valuation of the Equitable building in N. Y. City is \$12,400,000. What is the assessed valuation?

A variation of Type I occurs when instead of $12\,n = 132$ in the preceding Illustrative example, we have $8\,n + 4\,n = 132$. We can add $8\,n$ and $4\,n$ just as we add 8 dollars and 4 dollars to obtain 12 dollars or 8 oranges and 4 oranges to make a dozen oranges. The answer is $12\,n$.

Illustrative example I. Solve $5\,x + 3\,x + 4\,x = 36$.

Solution:
$$5\,x + 3\,x + 4\,x = 36.$$
$$12\,x = 36.$$
$$x = 3. \quad \textit{Ans.}$$

Check:
$$5 \cdot 3 + 3 \cdot 3 + 4 \cdot 3 = 36.$$
$$15 + 9 + 12 = 36.$$
$$36 = 36.$$

Illustrative example II:

Solve $9\,x + 2\,x - 8\,x = 48$.

Solution:
$$9\,x + 2\,x - 8\,x = 48.$$
$$3\,x = 48.$$
$$x = 16. \quad \textit{Answer.}$$

Check as above.

Exercises

Solve for x and check:

1. $3\,x + 2\,x + 4\,x = 45.$ **2.** $5\,x + 7\,x + 6\,x = 72.$

3. $25\,x + 45\,x + 10\,x = 32{,}000.$

4. $2x + 2\frac{1}{2}x = 90$.

Hint: $2x + 2\frac{1}{2}x = 4\frac{1}{2}x$.

5. $6\frac{1}{2}x + 1x = 75$.

6. $200 = 40x + 10x$.

7. $1x + .5x = 60$.

Hint: $1x + .5x = 1.5x$.

8. $1x + .45x + .55x = 8$.

9. $11x + 4x - 5x = 50$.

10. $32x + 4x + 6x - 5x - 2x = 700$.

11. $6.5x + 7x - .5x = 91$.

12. $9.5x + \frac{1}{4}x + \frac{1}{2}x - 1.25x = 153$.

Problems

1. One number is 3 times another, and their sum is 32. Find the numbers.

Solution: Let $1x$ and $3x$ represent the numbers.

$$1x + 3x = 32.$$

$$4x = 32.$$ Complete solution.

2. One number is 5 times another and their sum is 72 Find them.

3. One number is 8 times another and their sum is 90. Find them.

4. An automobile ran $25m$, $30m$, $28m$, and $36m$ miles in four consecutive days. If the total mileage was 1190 miles, how many miles did it travel each day?

5. Helen's savings for 4 consecutive months were $1c$ cents, $3c$ cents, $4c$ cents, and $7c$ cents. If her total savings were 150 cents, how much did she save each month?

6. Three numbers (sides of a triangle) are in the ratio of 3 to 4 to 5. The perimeter of the triangle is 96 inches. Find each side.

*Hint:** Let $3x$, $4x$, and $5x$ represent the numbers.

---

* $3x$ and $4x$ represent any two numbers in the ratio of 3 to 4. Similarly, $3x$, $4x$, and $5x$ represent any three numbers in the ratio 3 to 4 to 5.

7. The sides of a triangle are in the ratio of 2 to 3 to 4. The perimeter is 180 inches. How long is each side?

8. The ratio of two numbers is 6 to 4. Their difference is 20. Find them.

Hint: $6x - 4x = 20$.

9. Divide a line so that the parts shall be in the ratio of 3 to 2, and their difference shall be 20 inches. How long is each part?

10. Divide a line 48 inches long in the ratio of 2 to 1.

11. Divide \$3600 in the ratio 3 (3 to 1).

12. The ratio of the length to the width of a rectangle is 4 to 1. If the perimeter is 60 inches, find the dimensions. Draw figure.

13. Two angles whose sum is 180° have a ratio of 2 to 3. Find the number of degrees in each angle.

14. The ratio of two lines is 5 (5 to 1). Their total length is 66 inches. How long is each?

15. A number increased by 25% of itself is 25,625. Find the number.

Solution: Let x = the number.

$$1x + 0.25x = 25{,}625.$$

$$1.25x = 25{,}625.$$ Complete the solution.

16. What number increased by 24% of itself is 3720?

17. A building sold for \$16,000, which was 300% more than it cost. Find the cost.

18. A used car sold for \$630, which was 30% less than it cost. Find the cost.

TYPE II. $\frac{x}{a} = b$

Illustrative example:

Solve $\frac{x}{4} = 5.$

Solution: Think, what number divided by 4 gives 5? Therefore $x = 20$, or 4 times 5.

Check: $\frac{20}{4} = 5.$

$5 = 5.$

Exercises

Solve and check:

1. $\frac{x}{2} = 20.$
2. $\frac{x}{3} = 40.$
3. $90 = \frac{x}{8}.$
4. $\frac{x}{9} = 70.$
5. $\frac{x}{50} = 40.$
6. $9 = \frac{x}{100}.$
7. $\frac{x}{200} = 5.$
8. $4 = \frac{x}{25}.$
9. $\frac{x}{12\frac{1}{2}} = 8.$

Problems

Illustrative example:

What number divided by 6 gives 12?

Solution: Let n = the given number.

Then $\frac{n}{6} = 12.$

$n = 72.$

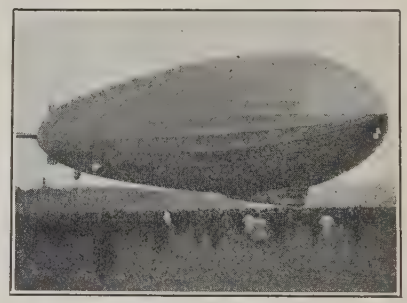
Check: $72 \div 6 = 12.$

1. What number divided by 12 gives 9?
2. A certain number divided by 20 gives 45. Find it.
3. What number divided by 65 gives 60?
4. John's earnings divided by 6 gives \$14.50. Find his earnings.
5. John's savings divided by 4 is \$5.45. Find his savings.
6. Mary's money divided by 3 is 15 cents. How much had she?

7. The total sales divided by 26 is \$52,000. Find total sales.

8. A firm's expenses divided by 16 is \$4800. Find their expenses.

9. The profit of a firm divided by 8 is \$5460.75. Find their profits.



THE GRAF ZEPPELIN

10. The distance traveled by the German Airship Zeppelin on its eastern trip to the U. S. divided by 5 was 1200 miles. How far did it travel?

TYPE III. $x + b = c$

Illustrative example:

Solve $1x + 6 = 24$ and check.

Solution: Think (cover up the $1x$ with your finger).

$$6 + ? = 24.$$
$$? = 18.$$

Therefore $x = 18.$

Check: $18 + 6 = 24.$
$24 = 24.$

Exercises

Solve for x and check. Work mentally when possible.

1. $1x + 5 = 10.$	**6.** $x + 12 = 15.$	**11.** $20 + x = 70.$
2. $1x + 7 = 19.$	**7.** $x + 15 = 35.$	**12.** $x + \frac{1}{2} = 9\frac{1}{2}.$
3. $1x + 9 = 21.$	**8.** $x + 25 = 50.$	**13.** $x + 5.2 = 17.2.$
4. $1x + 10 = 30.$	**9.** $60 = x + 40.$	**14.** $x + 7\frac{1}{2} = 12\frac{1}{2}.$
5. $1x + 8 = 17.$	**10.** $100 = x + 75.$	**15.** $x + 8.3 = 19.6.$

Problems

1. I am thinking of a number which increased by 11 is 45. Find the number.

2. By traveling with a wind blowing 40 miles an hour an airplane made 125 miles an hour. How fast could it fly in still air?

3. An owner increased the rent on a store \$20 a month. The new rent was \$135 a month. Find the old rent.

4. A shoe merchant advanced the price of a certain shoe \$1.50 a pair. The new price was \$8.50. Find the old price.

TYPE IV. $x - b = c$

Illustrative example:

Solve and check $1x - 12 = 8$?

Solution: Think (cover up the $1x$ with your finger), from what must I subtract 12 to obtain 8?

$$? = 20.$$

Therefore $x = 20.$

Check: $1 \cdot 20 - 12 = 8.$

$8 = 8.$

Exercises

Solve and check. Work mentally if possible.

1. $1x - 15 = 25.$
2. $1x - 40 = 70.$
3. $x - 9 = 50.$
4. $x - 12 = 2.$
5. $x - \frac{1}{2} = 10.$
6. $x - 2\frac{1}{2} = 20\frac{1}{2}.$
7. $x - 4.5 = 6.5.$
8. $x - 25 = 0.$
9. $x - 2\frac{3}{4} = 9\frac{3}{4}.$
10. $x - 200 = 800.$
11. $x - 65 = 135.$
12. $x - 2.5 = 12.5.$

Problems

1. Find a number such that if 20 is subtracted from it the result is 50.

2. After spending \$5.75 I had left \$22.30. How much had I at first?

3. After cutting off 23 inches from a stick it was 47 inches long. How long was it at first?

4. By deducting \$20 from a man's money for an automobile fine, he had \$75 left. How much had he at first?

5. After traveling 350 miles of my journey I still have 250 miles to go. How long was my journey?

Equations Solved by the Use of Two Types

Example 1. Solve $4x + 8 = 32$ and check.

Solution: Think $8 + ? = 32$.

$? = 24$.

Therefore $4x = 24$.

$x = 6$.

Check: $4 \cdot 6 + 8 = 32$.

$24 + 8 = 32$.

$32 = 32$.

Example 2. Solve and check $5x - 10 = 15$.

Solution: Think, from what number must I take 10 to get 15?

$? = 25$.

Therefore $5x = 25$.

$x = 5$. Check as above.

Exercises

Solve and check:

1. $5x + 4 = 34$.

2. $6x + 9 = 45$.

3. $2x + 3 = 19$.

4. $71 = 7x + 8$.

5. $\frac{1}{2}x + 4 = 12$.

6. $\frac{1}{4}x + 3 = 20$.

7. $1.5x + 4 = 22$.

8. $1.6x + 12 = 28$.

9. $7x - 12 = 9$.

10. $4x - 5 = 28$.

11. $\frac{1}{2}x - 4 = 6$.

12. $\frac{1}{3}x - 7 = 28$.

13. $28 = \frac{3}{4}x + 7$.

14. $\frac{5}{8}x + 9 = 44$.

15. $2.5x - 6 = 19$.

Problems

1. Six times a number added to 8 is 20. Find the number.

2. Twelve times a number plus 11 is 143. Find the number.

3. If 5 is subtracted from two times a number, the result is 59. Find the number.

4. If 10 cents is taken from 6 times my money, 62 cents remain. How much had I at first?

5. Five times my annual business expenses plus $2000 is $18,500. Find my annual business expense.

6. A tie and a collar together cost me $1.00. The tie cost me 50 cents more than the collar. Find the cost of each.

Practice Test in Solving Equations. All Types

Solve for x and check:

1. $7x = 56$.
2. $108 = 9x$.
3. $132 = 12x$.
4. $2x = 14\frac{4}{5}$.
5. $3x = 21\frac{9}{16}$.
6. $2x = 12\frac{1}{2}$.
7. $13\frac{1}{2}x = 27$.
8. $4\frac{1}{3}x = 13$.
9. $2.5x = 10$.
10. $5.5x = 55$.
11. $.07x = 42$.
12. $x + 5 = 10$.
13. $14 = 3x + 2$.
14. $6x + 2 = 20$.
15. $5x + 5 = 45$.
16. $3x - 1 = 8$.
17. $5 = 7x - 16$.
18. $15x - 5 = 25$.
19. $\frac{x}{9} = 4$.
20. $\frac{2x}{7} = 4$.

Selection Test

Select from the values given at the right, the one which you believe to be the correct answer for each of the following equations:

1. $3x = 16\frac{1}{2}$. — 5, $5\frac{1}{2}$, 6, $13\frac{1}{2}$.
2. $2x = 12\frac{1}{2}$. — 6, $6\frac{1}{2}$, $6\frac{1}{4}$, $10\frac{1}{2}$.
3. $6x = 66.6$. — 111, 11.1, 66, 60.6.

4.	$\frac{1}{2}x = 9\frac{1}{2}$.	9, 19, $4\frac{3}{4}$, 10.
5.	$\frac{1}{3}x = 3\frac{1}{3}$.	10, $1\frac{1}{9}$, 3, $3\frac{2}{3}$.
6.	$\frac{2}{3}x = 12$.	8, $11\frac{1}{3}$, $12\frac{2}{3}$, 18.
7.	$x + 2 = 6$.	4, 12, 3, 8.
8.	$x + 2\frac{1}{2} = 3\frac{1}{2}$.	6, 8, 1, $1\frac{1}{2}$.
9.	$x - 3 = 6\frac{1}{2}$.	9, $3\frac{1}{2}$, 10, $9\frac{1}{2}$.
10.	$3x + 6 = 12$.	4, 6, 2, 6.

Review

Write formulas or algebraic shorthand for:

1. The sum of the interior angles A, B, and C, of a triangle is 180°.

A Swimming Tank

2. The number of gallons (g) a rectangular swimming tank l feet long, w feet wide, and h feet high will hold if 1 cubic foot holds approximately 7.5 gallons.

3. 3500 if $M = 1000$ and $C = 100$.

4. The total wages (T) is the wage (w) per hour multiplied by the number of hours (h).

If the number of hours increases, then the wages . . .

5. Distance (d) is the product (result of multiplication) of the rate (r) by the time (t).

6. Kinetic energy (in physics) (K) of a falling body is the product of the weight (w) in pounds by the height (h) that it falls.

7. The number of tons (T) of hard coal in a rectangular bin is found by dividing the product of the three dimensions, l, w, and h, in feet by 35.

8. The altitude of a triangle is twice the area (A) divided by the base (b).

9. The total cost C for a telegram of over 10 words is 25 cents for the first 10 words and 2 cents for each additional word w.

10. The cost C of any number of ounces of letters by airmail at 5 cents for each ounce o.

11. A number N greater than 10 and less then 100 is 10 times the tens' digit t added to the units' digit u. Write 68 in terms of t and u.

12. Six times a number increased by 5 equals 23.

13. A number added to 5 times itself gives 24.

14. The length of a rectangle which is 6 feet longer than the width w.

15. Find the cost c per square foot for a canvas tarpaulin to cover the football field at the Yankee Stadium in N. Y. City if the total cost is \$10,000 and the canvas is 360 feet by 160 feet.

Complete the following:

16. The average rate per hour that you make in going 100 miles depends upon . . .

17. The area of a triangle whose altitude is 20 inches depends upon . . .

18. My chances of getting into college will depend upon . . .

19. The amount that a man earns at \$6 a day depends upon . . .

20. The amount of money that a real estate agent earns, if he is paid by commission only, depends upon . . .

21. The number of tons of coal that I can buy for \$100 depends upon . . .

22. In the formula $C = np$, if $n = 10$ and p increases, then C . . . If p decreases, then C . . .

CHAPTER III

NEGATIVE NUMBERS. FUNDAMENTAL OPERATIONS

In the preceding chapter we saw that algebra differs from arithmetic in using letters as well as figures. This use of letters enables us to make formulas and equations and so to solve problems more easily than we could by arithmetic alone. In fact, the greatest value of algebra is in solving problems by using equations of the types studied in the last chapter.

17. Negative Numbers. — As we proceed with the examples given, and prepare for further use of equations we shall find our work greatly simplified by the use of a new kind of number. In arithmetic all numbers are positive; algebra uses also *negative numbers*.

In arithmetic we subtract any number from another number which is equal or greater, such as:

	6	5	4	minuend
(a)	4	4	4	subtrahend
	2	1	0	remainder

But subtraction like the following:

	3	2	1	0	minuend
(b)	4	4	4	4	subtrahend
	?	?	?	?	

does not appear in arithmetic. We cannot subtract a larger number from a smaller number in arithmetic.

Now notice in (a) above, that as the minuend decreases, the remainder also decreases, until the minuend and subtrahend are equal when we obtain 0 for our remainder.

Now note in (b) above that our minuend is less than the subtrahend, therefore our remainder must be smaller than 0.

Hence: The subtracting of a greater number from a smaller number is possible only by thinking of numbers smaller than 0.

A *negative number* is an invention which permits us to subtract b from a in $a - b$ when b is larger than a. It is a number preceded by the $-$ sign. It is the opposite of a positive number.

18. Signed Numbers. — Positive numbers have no sign at all, or sometimes a $+$ sign. Negative numbers must be preceded by a $-$ sign. Thus we see that the signs plus $(+)$ and minus or negative $(-)$ have two uses each. When they indicate addition or subtraction, as in solving problems, they are *signs of operation*. When they denote positive or negative quality, as when standing alone, they are called *signs of quality*.

If the sign precedes a number which stands alone, or if it is associated with a number in a parenthesis, it is a sign of quality; if it stands between two numbers, it is usually a sign of operation.

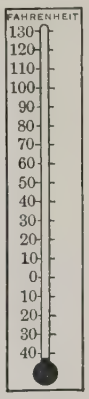
Thus: $(-a) \overset{\text{o}}{+} b \overset{\text{o}}{+} (-c) \overset{\text{o}}{-} d$, the signs marked o above them are signs of operation. This means that $-a$, b, and $-c$ are to be added, and d subtracted from their sum.

Note that when no sign precedes a number it is considered positive.

Signed numbers are numbers preceded by $+$ or $-$ signs, in contrast with the unsigned numbers of arithmetic.

19. Absolute Value. — In algebra the absolute value of a number means the number of units it contains regardless of its sign. Thus, the absolute value of either $+10$ or -10 is 10, since there are ten units in either number.

20. Meaning of Negative Numbers. — We have seen thermometers, and we know that a certain point on them is marked 0, and called zero. If we read in the paper some winter morning that the lowest temperature recorded on the preceding day was − 10 degrees, what should we understand that statement to mean?



By a similar method how should we indicate 5 degrees above zero?

If distances north of a certain point are considered positive, then distances south of that point would be negative; if distances up are considered positive, distances down are negative; if distances east are considered positive, distances west are negative.

Dates are reckoned in Christian countries from the birth of Christ. Dates since Christ (A.D.) may be considered positive, and those before Christ (B.C.) negative.

Numbers representing assets, that is, actual property owned, may be considered positive; and liabilities, that is, debts or obligations, which are the opposite of property owned, may be considered negative.

From these illustrations we see that *positive and negative numbers are opposites of each other in time, direction, or character.* They start in opposite directions from the point zero.

21. Number Scale. — This may be illustrated by a diagram showing the relation of small positive and negative numbers. Below is such a scale.

$$-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5 \quad +6$$

Let a line of indefinite length be drawn and a certain point on it be taken to represent 0. Then equal distances are laid off on the line in both directions from the point. The

points at the right of zero are numbered successively $+1$, $+2$, $+3$, . . ., while those at the left are numbered -1, -2, -3, etc. Thus we see that every positive or negative number might be represented by some point on this line. Positive and negative numbers may be thought of as *directed numbers*.

Exercises

1. Julius Caesar died in 44 B.C., Napoleon in 1821 A.D., and George Washington in 1799 A.D. Represent each of these dates by appropriate positive or negative numbers.

Pike's Peak

2. Elevations are reckoned from sea level. The elevation of Pike's Peak is 14,108 feet above sea level, while that of the Dead Sea is 1292 feet below sea level. Represent these elevations by numbers with appropriate signs.

Represent each of the following by signed numbers:

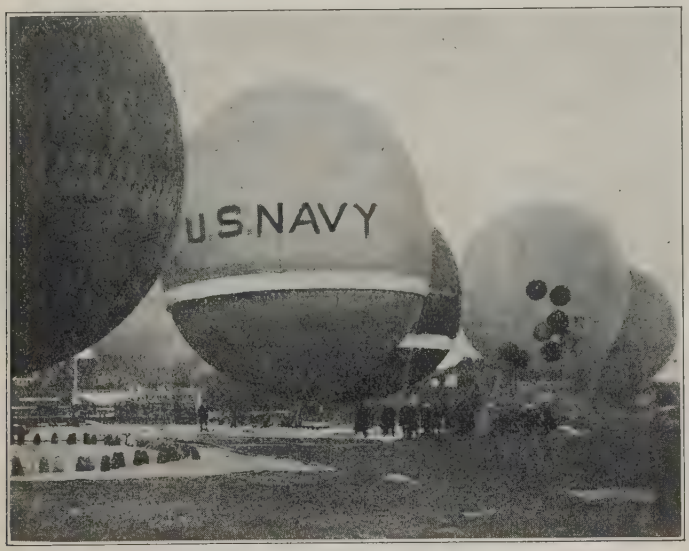
3. 48° North Latitude, $82\frac{1}{2}$° North Latitude, 67° South Latitude.

4. 120° West Longitude, 62° East Longitude, 10° East Longitude.

5. The top of a coal breaker was 110 feet above the surface of the ground. The first mine level was 86 feet below

the surface, while the second mine level was 180 feet below it. Represent this by signed numbers.

6. If a thermometer registers 10° above zero at noon and falls 20° by midnight, what represents the temperature at midnight?



PREPARING FOR A BALLOON RACE

The upward pull of the balloon is negatived by the attached weights.

7. If a balloon lifts up with a force of 250 pounds, how much weight must be attached to it to keep the balloon from rising? If we call the lifting force +, represent the weight.

8. A ship has a displacement of 30,000 tons (*i.e.* sinks into the water until it displaces 30,000 tons of water). Express the weight of the ship and the upward pressure of the water by + and − signs.

9. Mr. Jones owns a house and lot worth $7000, furniture $1200, Liberty bonds $400, and has $850 in the bank.

He owes Mr. Brown \$250, and Mr. Smith \$425. Represent each of these amounts by a number with the proper sign.

10. See if you can discover how to add the numbers in problem 9 in order to find how much Mr. Jones is actually worth.

22. Negative Numbers in Graphs. — In our study of graphs we thought of them as representing only positive numbers, but frequently some of the numbers in a graph are negative. A temperature graph is a simple illustration of this. In winter, temperatures frequently fall below zero, thus becoming negative. In such cases the *origin* should be taken to represent zero. *Positive* temperatures should be marked off on the vertical axis *above* the origin, and *negative* numbers *below* the origin. After the points are located, the graph is drawn as usual.

Illustrative example I:

The temperature on a winter day, taken every 2 hours for 24 hours, shows the following:

12 Midnight	− 2°	10 A.M. . .	25°	8 P.M. . .	10°
2 A.M. . .	− 4°	12 Noon . .	28°	10 P.M. . .	2°
4 A.M. . .	− 4°	2 P.M. . .	33°	12 Midnight	− 2°
6 A.M. . .	10°	4 P.M. . .	30°		
8 A.M. . .	20°	6 P.M. . .	22°		

Interpretation.

By study of the graph answer the following questions:

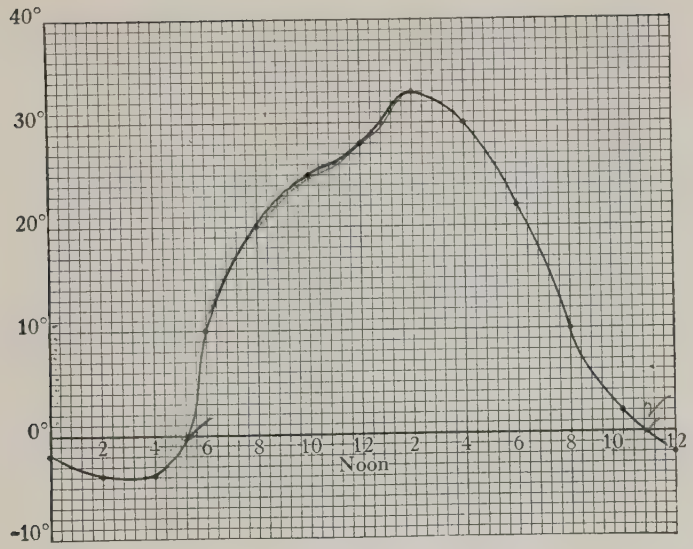
What was the lowest temperature of the day?

At what hour did it occur?

Did it thaw at any time during that day? If so, when?

At what times was the temperature just zero?

How much higher was the temperature at midnight than at 2 A.M.? At 6 A.M.? At 4 A.M.?



Illustrative example II:

The following elevation readings were taken on a straight line at right angles to a river. The water level is considered 0, elevations above the water level are indicated by the + sign, and those below the water level by the − sign. The readings were taken at stations 20 feet apart.

Station	Elevation	Station	Elevation	Station	Elevation
1.	+ 28	5.	− 12	9.	− 6
2.	+ 6	6.	− 8	10.	0
3.	− 2	7.	− 4	11.	4
4.	− 15	8.	− 1	12.	6

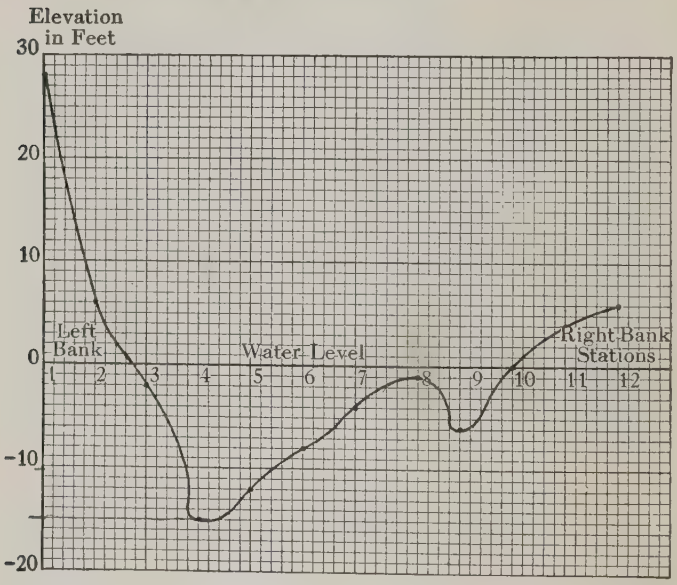
Interpretation.

By a study of the graph answer the following questions:

Which is the steeper bank?

Nearer which bank is the main channel of the river?

What is its greatest depth? Locate a shoal.
How would an island appear in a graph of this kind?



Exercises

1. Make a graph to show the following temperature readings for a winter day:

6 P.M.	4°	4 A.M.	− 8°	2 P.M.	4°
8 P.M.	0°	6 A.M.	− 10°	4 P.M.	10°
10 P.M.	0°	8 A.M.	− 2°	6 P.M.	15°
12 Midnight	− 2°	10 A.M.	4°		
2 A.M.	− 6°	12 Noon	0°		

2. The following readings, taken at intervals of 10 feet, show the elevations across the valley of a shallow stream. Draw a graph showing the valley in cross section.

1. + 40	6. 0	11. + 1
2. + 32	7. − 4	12. + 5
3. + 38	8. − 2	13. + 8
4. + 20	9. − 3	14. + 12
5. + 4	10. − 1	15. + 20

Answer similar questions to those in the illustrative example.

3. An engineer wished a curve graph to show the depth of a stream. He found the depths from the east side to the west side at intervals of 10 feet to be as follows (13′ = 13 ft.): W. 0, 8′, 13′, 22′, 32′, 18′, 21′, 28′, 42′, 27′, 17′, 11′, 5′, 0′. E.

23. Formula Graphs Using Negative Numbers. — In the graphs so far discussed we have usually marked off equal spaces on the horizontal axis to represent equal intervals of time, and on the vertical axis equal spaces to represent equal changes in the other quantity, *the value of which changed as the time changed.* Such a changing quantity is called a *variable.*

Our former formula graphs represented positive numbers, all of which were above the horizontal axis. We are now going to graph negative numbers as well, and as these negative numbers come below the horizontal axis, it is important to place this axis well above the bottom of the squared paper.

In the formula $C = \frac{5}{9}(F - 32)$, which expresses algebraically the relation between the Centigrade and Fahrenheit thermometers, C changes as F changes; the value of C depends on the value of F. We may, therefore, graph the formula as follows:

Make a table of values as in former graphs, and on the horizontal axis mark off to the right of the origin equal spaces to represent successive values of F, and on the vertical axis

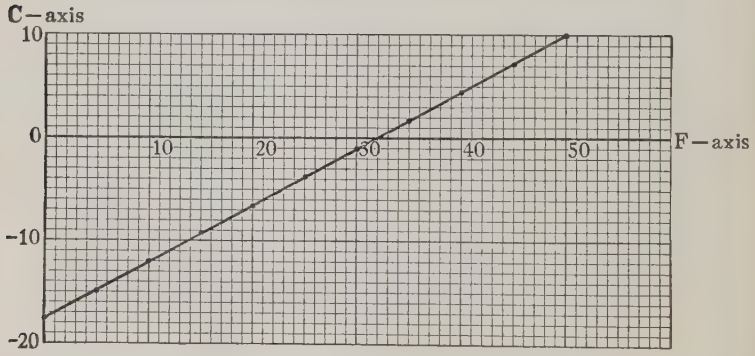
similar equal spaces for values of C. We shall now call these axes the F-axis and the C-axis, respectively.

If, now, $F = 0$, then $C = \frac{5}{9}(-32) = -\frac{160}{9} = -17\frac{7}{9}$.

Similarly, we may substitute other values for F, and obtain corresponding values of C, as shown in the following table:

F	0	5	10	15	20	25	30	35	40	45	50
C	$-17\frac{7}{9}$	-15	$-12\frac{2}{9}$	$-9\frac{4}{9}$	$-6\frac{2}{3}$	$-3\frac{8}{9}$	$-1\frac{1}{9}$	$1\frac{2}{3}$	$4\frac{4}{9}$	$7\frac{2}{9}$	10

We now locate the points as explained on page 33.



The graph pictures the relation between the same temperatures as recorded by a Centigrade thermometer (C) and as recorded by a Fahrenheit thermometer (F).

Interpretation.

What temperature Centigrade corresponds to 7 degrees Fahrenheit?

Solution: Find the mark on the F-axis which corresponds to 7 degrees. Follow the vertical line downward until it intersects the graph. Then follow the horizontal line from the point of intersection until it meets the C-axis. Here we read

-- 14 degrees, which is the desired result. Hence, 7 degrees Fahrenheit equals approximately − 14 degrees Centigrade.

In like manner change to Centigrade scale the following Fahrenheit readings: 4 degrees; 13 degrees; 22 degrees; 32 degrees; 42 degrees; − 1 degree.

Change the following Centigrade readings to the Fahrenheit scale: − 12 degrees; − 8 degrees; 5 degrees; 2 degrees. How does the Centigrade reading change as the temperature falls from 212° to 32° Fahrenheit? Extend graph.

24. Addition of Signed Numbers by a Scale. — Now that we have illustrated and interpreted negative numbers, we are ready to add, subtract, multiply, and divide them. Addition is best shown by a scale like the Number Scale in section 21. Let us draw a straight line of indefinite length and mark a certain point on it 0. Then mark off equal divisions in both directions from 0, numbering the divisions to the *right*, with the *positive* integers and those to the *left*, with *negative* integers, thus:

. . . . . .

−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10

To add 4 to + 3, begin at + 3 and count 4 spaces to the right. What is your answer? In the same way we may use this scale to add + 6 to − 4. We first locate − 4 on the scale and count to the *right* 6 units, which brings us to + 2. This is the sum.

If we were to add to − 4 a negative number as − 2, we should begin at − 4 and count 2 units to the *left*, giving us as the answer − 6. These illustrations lead to the following

Rule. — To add two signed numbers by a scale, *locate the first number on the scale. From it count as many units as there are in the second number, to the right if it is positive, and to the left if it is negative. The number reached is the sum.*

Exercises

Using the above scale add the following:

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
+5	−4	−7	+7	+8	+7	−2	−6	+2	−2
+4	+8	+10	−5	−9	−7	−5	−4	−11	+11

11.	12.	13.	14.	15.	16.	17.	18.	19.	20.
−6	−10	+10	+4	+7	−9	0	0	0	−9
−3	+7	−7	−3	−9	+7	−6	+7	−9	+9

25. Algebraic Addition of Signed Numbers. — In section 20 we learned how positive and negative numbers might be represented by assets and liabilities respectively. By the use of this illustration we shall develop a rule for the addition of such numbers algebraically.

(a) If Henry has 20 cents in his pocket, this sum is represented by + 20. If he owes Frank 8 cents, this debt is represented by − 8. Then Henry is really worth only 12 cents since 8 cents must be used to pay his debt. This shows that the combination or addition of + 20 with a − 8 is equal to + 12.

(b) If Henry's debt were 30 cents while he had only 20 cents in money, he would really be worth 10 cents less than nothing since he would still owe 10 cents after applying the money he had toward his debt. This illustrates the fact that + 20 added to − 30 gives as a result − 10.

(c) If Henry has 10 cents in one pocket and 5 cents in the other, how much does he have in all?

Therefore, $(+10)+(+5)=\ ?$

(d) If Henry owes Frank 10 cents and John 15 cents, how much does he owe in all?

Therefore, $(-10)+(-15)=\ ?$

If we were to set these problems down as sums, we should then have the following:

+ 20	− 30	+ 10	− 10
− 8	+ 20	+ 5	− 15
+ 12	− 10	+ 15	− 25

By studying these results we are able to state the following rule for the addition of signed numbers:

Rule. — *To add two numbers having like signs, find the sum of their absolute values* (section 19) *and prefix their common sign.*

To add two numbers having unlike signs, find the difference between their absolute values and prefix the sign of the number having the greater absolute value.

Exercises

1. Add each of the following, remembering that an unsigned number is positive:

− 1	+ 5	− 3	5	− 17	12	14	− 15
+ 3	+ 2	− 6	− 1	11	− 10	22	− 8
− 12	+ 15	− 10	+ 2	6	7	− 7	− 10
+ 4	− 3	− 2	− 11	− 5	− 1	+ 3	8

2. (a) 6 dollars
5 dollars

(b) 6
5

6. 6
− 2

3. (a) 6 miles
− 7 miles

(b) + 6
− 7

7. 7
− 2

4. (a) − 4 cents
6 cents

(b) − 4
+ 6

8. − 5
3

5. (a) − 6 dimes
8 dimes

(b) − 6
+ 8

9. 7
− 3

10. − 12 bushels
15 bushels

11. − 5 yards
− 6 yards

***12.** 5 dollars
6 dollars
− 3 dollars

13. (− 3)+(+ 5)+(+ 7).

14. 24 degrees
16 degrees
− 10 degrees
+ 20 degrees
− 30 degrees

15. Collect* (add): 10 +(− 3); 8 +(− 2); 6 + 3; (− 10)+ 2; 8 +(− 3); − 18 +(− 6).

16.

− 10	8	− 6	+ 9 degrees
+ 2	− 10	+ 8	− 5 degrees

17.

− 5	+ 5	8	6
− 10	− 3	− 6	− 11

18. − 10 +(− 3)+(+ 8)+ 10;
+ 10 + 3 +(− 8)+(− 5)+ 1;
− 8 +(− 2)+ 3 +(− 1)+ 7 +(− 10)+(− 4).

19.

10	8	− 7	+ 6	− 10 cents	+ 5 dollars
− 9	6	+ 5	+ 4	− 12 cents	+ 7 dollars

20.

+ 16	+ 25	− 15	− 12	+ 11
− 15	− 24	+ 13	+ 10	− 9
+ 12	+ 11	− 4	+ 3	+ 5

21. (+ 5)+(− 10)+(− 8)+(− 6)+(+ 7)= ?
(− 12)+(− 11)+(− 10)+(− 8)+(+ 40)= ?
(+ 14)+(+ 11)+(+ 12)+(+ 9)+(− 45)= ?

* Addition may be performed in any order. The student may combine the numbers in the order in which they occur, or, if he prefers, he may combine the positive numbers, then combine the negative numbers, then combine the results. "Collect" is another expression for "add."

22.

-4 eighths	$+5$ sixths	-3 twelfths	$+15$ sixteenths
-7 eighths	-4 sixths	$+7$ twelfths	-13 sixteenths
$+6$ eighths	$+3$ sixths	$+9$ twelfths	-12 sixteenths

23.

$+5$	-5.4	$+6.3$	$+7.4$	$\frac{1}{3}$
-6	$+3.5$	-7.4	$+6.3$	$\frac{1}{2}$
-7	-4.6	$+8.5$	$+5.2$	$\frac{1}{4}$

24. (a) $6+(-4)$.
(b) $7+(-6)$.
(c) $10+(-9)$.
(d) $(-6)+(-4)$.
(e) $(-7)+(+9)$.
(f) $(+9)+(-12)$.
(g) $(-5)+(-3)$.
(h) $(-2)+(-6)$.
(i) $(-11)+(+15)$.

Combine (add):

25. $+2, +6, -4, -3, +7, -9, -11, +5$.

26. $-3, 5, 11, -2, -1, 12, 6, 8, -4, 2, -10, +12$.

27. $-5, 6, -10, 2, 4, +10, -8, -3, +5, -7$.

28. $+7, -8, -2, +9, -8, -5, +4, +10, -12, +15, +6, -14, -3, +20$.

29. $-9, -7, -6, +5, +7, +8, -7, +10, -20, -15, +25, +6, -30, -40$.

30. $-2, +5, -6, +3, -7, -2, +9, +10, -5$.

31. $2, +4, -6, -4, +7, -2, +9, -7$.

32. $-5, -7, +4, +8, -8, -2$.

33. $-4, -6, +7, -3, +5, -8, -2, +4$.

34. $5, -6, +2, -4, -3, +5, -8, -2, +4$.

35. $-6, -8, -7, +9, -11, -5, +10, -4$.

36. $+600, -400, -200, -300, +500, +400, +200$.

37. $-250, -350, +500, +700, -900, +400$.

38. $4, -5, +7, +9, -3, +5$.

For other work in addition see page 90.

Interpreting Word Problems

If you go 15 miles east in your automobile, then go 20 miles east from that point, you are then 35 miles east from your starting point, that is, $+15+(+20)=+35$. If you go 20 miles east and then 5 miles west, you are $+20+(-5)=+15$ miles east of where you started.

Using signed numbers, interpret each of the following, starting from a point that we shall call 0 in each case.

1. Five miles north, then 10 miles north.
2. Six miles west, then 17 miles west.
3. Eight miles south, then 12 miles south.
4. Five miles east, then 20 miles east.
5. Fifteen miles east, then 5 miles west (opposite).
6. Four blocks north, then 8 blocks north.
7. Nine blocks north, then 8 blocks north, then 3 blocks south.
8. 3 miles up in an airplane, then dropping 1 mile down.
9. 18,000 feet up in an airplane, then dropping 4000 feet.
10. Going up 20 stories in an elevator, then coming down 5 stories.
11. Going up 30 stories and then coming down 10 stories.
12. Going down 15 stories and then going up 20 stories.
13. Going north 20 blocks, then walking back 15 blocks.
14. Going east 7 blocks, then walking back 5 blocks.
15. Going down 9 floors in an elevator, then up 15 floors.
16. Gaining 6 pounds in weight and then gaining 5 pounds more.
17. Gaining 7 pounds in weight and then losing 9 pounds.
18. Going 60 miles north in an automobile, then returning 60 miles south.

19. Going 10 yards, then 5 yards in a football game.

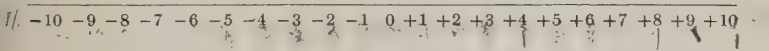
20. Gaining 25 yards and being put back 10 yards by the referee for holding in a football game.

21. If you run a motor boat up a stream at the rate of 30 miles per hour and the current of the stream is 5 miles per hour, what is your actual speed up that stream?

26. Subtraction of Signed Numbers. — If you make a purchase for 68 cents, and hand the clerk a dollar bill, he may hand you in succession 2 cents, 5 cents, and 25 cents, saying as he did so "70 cents, 75 cents, a dollar." This method of making change is familiar to every boy and girl. The clerk has found the difference between 68 cents and a dollar by determining what must be added to 68 cents to make a dollar.

(a) What number added to 6 makes 8? Then $8 - 6 = ?$
(b) What number added to 0 makes 8? $8 - 0 = ?$
(c) What number added to -2 makes 8? $8 - (-2) = ?$
(d) What number added to -8 makes 8? $8 - (-8) = ?$
(e) What number added to 4 makes -3? $-3 - (+4) = ?$

Subtraction by the use of a scale.



From $+8$ take $+3$ by counting off spaces from $+3$ to $+8$. Which way on the scale do you count? The answer is $+5$.

Rule. — *To subtract one signed number from another.*

1. *Begin on the scale at the number to be subtracted* (the subtrahend).

2. *Count to the minuend.*

3. *The number of spaces counted is the result, positive if you count to the right and negative if you count to the left.*

Subtract each of the following as above:

+8	+8	+8	0	−1	+4	+2	0	−1	−3
+7	+8	+10	+4	+4	−3	−3	−3	−3	+3

How does the remainder change

(*a*) As the subtrahend increases if the minuend remains unchanged?

(*b*) As the minuend decreases if the subtrahend remains unchanged?

Exercises

1. Answer each of the following by supplying the proper number for the question mark. The first question should read "What number added to −5 will produce 8?" etc.

−5	6	3	4	10	−6	10	0	5
?	?	?	?	?	?	?	?	?
8	11	0	−2	3	−2	+6	−10	1

2. By referring to your answers to the above examples subtract the lower number from the upper number; then try it on the scale and see that it checks.

8	11	0	−2	3	−2	6	−10	1
−5	6	3	4	10	−6	10	0	5

3. Subtract

6	10	−8	−5	12	0	−6	0	−6
−3	2	6	−1	3	−8	−10	12	+3

−6	−3	+4	+2	−6	+2	−8	−6	+2
+2	+6	−10	−11	+5	+6	+10	−9	−7

0	−15	−8	+6	−9	+4	−3	+13
−3	9	−28	−5	−2	+11	+8	−5

27. Changing the Sign. — Let us subtract -8 from 6 by the above methods. We naturally ask "What number added to -8 will produce 6?" Or we may count on the scale. The answer in either case will be found to be 14. In like manner we obtain:

10	-3	12	-2
-3	-10	4	7
13	$+7$	8	-9

How do these results compare with the addition of the following:

10	-3	12	-2
$+3$	$+10$	-4	-7

A study of these results shows that we should get the same results if we had changed the sign of the subtrahend in each case and then had added according to the rule in section 25. Subtracting a negative * number is the same as adding a positive number of equal absolute value (*i.e.* subtracting a -4 produces the same result as adding a $+4$), or $-(-4)=+(+4)$. So also subtracting a positive number produces the same result as adding a negative number of the same absolute value, $-(+4)=+(-4)$.

We may then, as a short process, subtract by means of the following

Rule. — *To subtract signed numbers, mentally change the sign of the subtrahend and proceed as in algebraic addition.*

NOTE. Do not forget that algebraic addition may involve arithmetical subtraction (see rule, section 25, second part), and that algebraic subtraction may involve arithmetical addition.

The reasonableness of the above rule may be further shown by reference to our former illustration of assets and liabilities

* Some teachers may prefer the use of the word *minus* at times for negative numbers, or *plus* for positive numbers.

as representing positive and negative numbers. If some one should rob you of \$5, this would be a case of subtracting + \$5 from your possessions. If your grocer hands you an overdue bill for \$5, there would be added to your possessions − \$5. The result is the same, since you would be \$5 poorer in each case.

Similarly, if some one gave you \$5, there would be added to your possessions + \$5, while, if instead, he paid for you a debt of \$5, there would be subtracted from your possessions − \$5. In either case you would be made \$5 wealthier thereby.

So we see that adding a negative number is equivalent to subtracting a positive number of equal absolute value, and vice versa. Thus, such an expression as $5-(-3)+(-6)$ may be more simply written $5+3-6$, since subtracting a -3 is the same as adding a $+3$, and adding a -6 is the same as subtracting a $+6$.

Exercises

1. Subtract and check by adding your result to the subtrahend to see if it produces the given minuend.

− 10	8	10	6	− 4	− 4	+ 4	6	10
3	6	− 11	− 3	− 4	+ 4	− 4	− 13	6
− 4	− 5	6	5	− 3	− 8	− 3	− 10	− 6
+ 10	− 12	− 2	− 4	+ 6	+ 9	+ 3	− 15	− 3

2.

− 6	− 11	− 8	+ 6	+ 3	− 6	23	11
− 5	+ 3	− 10	− 5	+ 10	18	− 15	6

3.

\$5	− \$9	12	− 6	+ 8	− 2	+ 12	− 3	− 5	+ 9	+ 23
\$3	+ \$4	− 5	− 6	+ 2	− 5	− 4	− 7	+ 4	− 6	− 4

4.

+ 10	+ 9	− 13	− 25	+ 7	+ 12	+ 17	− 4	+ 9	+ 8
+ 6	+ 11	− 16	+ 6	− 3	+ 5	+ 8	+ 7	− 7	− 4

5. $+15+27+3-8+18+20+30+15+35+5$
$+16+4-3+8-6+50-20-10-3-3$

6. What number added to -3 will produce -10? -1? 6? 15? 0?

7. What number added to 8 will produce 12? 4? 0? -10? -1?

8. What number subtracted from -5 will produce -7? -12? -1? 0? 10?

9. What number subtracted from 12 will produce 0? 11? -20? 4? -2? 7?

10. What number subtracted from $12\,y$ gives $5\,y$?

11. Subtract $15\,x$ from $-25\,x$.

Subtract and check:

12.

841	24	− 51	− 176	276	234
− 311	− 372	− 197	+ 18	141	807

13.

− 5.25	+ .45	− .3 ft.	− 14.6 lb.	+ 45.6%
+ 4.65	− 3.23	− 1.5 ft.	− 16.8 lb.	+ 23.8%

14.

$+2\frac{1}{2}$	$+4\frac{5}{8}$	$+\frac{1}{2}$	$-15\frac{1}{2}$
$+3$	$-3\frac{1}{2}$	$-3\frac{3}{4}$	$+4\frac{3}{4}$

15. What is meant by $-4-(-3)$? Find the value.

Hint: Never say 3 minus for minus 3.

Find the value of the following:

16. $6-(-6)$; $-4-(-9)$; $7-(+9)$; $-10-(-12)$; $8-(-7)$; $-6-(+7)$.

17. If a thermometer registered $-12°$ one day, and the next day $-7°$, what was the change in temperature?

18. Find the difference in time between the year 25 A.D. and 30 B.C.

Looking Along the Chicago River

19. What is the difference in longitude between Chicago $(+ 88°)$ and Berlin $(- 13°)$?

20. Augustus Cæsar lived from $- 60$ to $+ 14$. How old was he when he died?

Further work in subtraction on page 97.

28. Multiplication. — In arithmetic we have been accustomed to regard multiplication as a short process of addition. For example, 4 times 5 is equivalent to $5 + 5 + 5 + 5$. Similarly, in algebra we may regard multiplication by a positive number as a short process of addition, while multiplication by a negative number is a short process of subtraction. Thus:

(a) $(+ 4) \times (+ 3) = + 3 + 3 + 3 + 3 = + 12.$

(b) $(+ 4) \times (- 3) = (- 3) + (- 3) + (- 3) + (- 3) = - 12.$

(c) $(-4)\times(+3)=-(+3)-(+3)-(+3)-(+3)=-12.$

(d) $(-4)\times(-3)=-(-3)-(-3)-(-3)-(-3)=+12.$

What is the sign of the result if two positive numbers are multiplied? If two negative numbers? If one positive and one negative? What, then, is the product of -3 and $+5$? Of 6 and -4? Of -5 and -2? Of -8 and 2? Of $+9$ and $+3$? Of $+8$ and -5? What is the value of $(-2)^*(-6)$? $3(-11)$? $(-6)(10)$? $+6(3)$? $-7\cdot 5 = ?$ $7\cdot 7 = ?$ $(-2)7 = ?$ $(-3)(-9) = ?$ $2(-5) = ?$

By inspection of the above results we infer the following

Rule. — *When two numbers having like signs are multiplied, the sign of the product is +; when two numbers having unlike signs are multiplied, the sign of the product is −.*

In connection with multiplication we must remember that xy means $x\cdot y$, while 24 means $20+4$. This is one of the chief differences between the notation of algebra and that of arithmetic.

Exercises

Multiply each of the following:

1. $3\cdot 4$; $3(-2)$; $5(-8)$; $-3(6)$; $-2(10)$.

2. $(-4)(-8)$; $(+3)(-6)$; $(-7)(+5)$; $(-8)(-3)$; $(-7)(6)$.

3. $(-3)(-5)$; $2(-7\frac{1}{2})$; $(-3\frac{1}{3})(-3)$; $2\frac{1}{2}(-4)$; $(-4)(-6\frac{1}{4})$.

4. $(-5)(-10)$; $4(-8)$; $-6(\frac{1}{2})$; $3(\frac{2}{3})$; $-10(-\frac{1}{2})$.

5. $3\cdot 5(-2)$; $6(-3)(4)$; $(-5)(-3)(2)$; $(-4)(3)(-1)$.

6. $(-3)(-2)(-6)$; $2(3)(-8)$; $5\cdot 4\cdot 10$; $3\cdot 8(-6)$.

7. $(-3)(-5)(-10)$; $3(\frac{2}{3})(-\frac{5}{2})$; $4(-6)(-2)(-1)$; $(-2)6(-5)$.

* Remember that in algebra the lack of a sign between two quantities means that they are to be multiplied.

29. Division the Opposite of Multiplication. — If the product of two numbers is divided by one of the numbers, the quotient is the other number. Thus, since $2 \times 5 = 10$, $\frac{10}{5} = 2$, or $\frac{10}{2} = 5$.

By the rule of multiplication stated in the preceding section

(a) $(+4) \cdot (+7) = +28$, therefore, $\frac{+28}{+7} = ?$

(b) $(+4) \cdot (-7) = -28$, therefore, $\frac{-28}{-7} = ?$

(c) $(-4) \cdot (+7) = -28$, therefore, $\frac{-28}{+7} = ?$

(d) $(-4) \cdot (-7) = +28$, therefore, $\frac{+28}{-7} = ?$

From the above illustrations we infer the following

Rule. — *The quotient of two numbers having like signs is positive, while the quotient of two numbers having unlike signs is negative.*

NOTE. The single exception to the above principle is found when one of the factors of the product is 0. For example, $5 \times 0 = 0$, but we cannot say that $\frac{0}{0} = 5$, since we could just as well write $10 \times 0 = 0$ and then infer that $\frac{0}{0} = 10$. Thus we cannot determine the value of $\frac{0}{0}$, so it is called *indeterminate*. It may have any finite value. *Division by* **0** *is not permissible.*

Exercises

Perform the following operations:

1. $6 \div 3$; $-6 \div 3$; $-6 \div (-3)$; $(+6) \div (-3)$.

2. $(-8) \div 2$; $14 \div (-7)$; $-12 \div 4$; $-10 \div (2\frac{1}{2})$.

3. $24 \div (-6)$; $(-16) \div (-8)$; $25 \div (-5)$; $(-18) \div (-9)$.

4. $6.25 \div (-5)$; $(-12.5) \div (-5)$; $(-7.5) \div 1.5$.

5. $\frac{20}{-4}$; $\frac{-32}{8}$; $\frac{-50}{-10}$; $\frac{18}{2}$; $\frac{-27}{3}$; $\frac{-49}{-7}$.

6. $\frac{0}{2}$; $\frac{0}{-5}$; $\frac{-6}{-1}$; $\frac{-12}{-6}$; $\frac{28}{-7}$; $\frac{-21}{3}$.

7. Divide $+8$ by $+4$; $+8$ by (-4); -8 by $(+4)$; -8 by (-4).
$+16$ by (-8); -16 by (-8); -16 by $(+8)$.
-20 by (-5); -24 by $(+6)$.

8. $\frac{24}{-4}$; $\frac{-12}{-6}$; $\frac{-15}{3}$; $\frac{32}{8}$; $\frac{+48}{-12}$; $\frac{10}{-2}$.

9. $8(-4) \div 16$; $(-6)(-10) \div 15$; $3(-30) \div (-6)$.

10. $8 \div 4 \cdot 5$; $(-6)(-5) \div 2$; $(-15) \div (-3) \cdot (-8)$; $(-7)(-14) \div (-2) \cdot 3$.

Review

1. Represent by appropriate signed numbers the following distances north and south from New York: 180 miles north, 10 miles south, 20 miles south, 32 miles north.

2. The elevations of ten points in southeastern Russia are: $+80$ ft., $+104$ ft., -32 ft., -10 ft., -60 ft., 0, $+204$ ft., $+96$ ft., $+38$ ft., -16 ft. Find the average elevation for this region.

3. Add:

$+16$	$+32$	-20	-18	$+40$
-16	-15	$+4$	-20	$+11$

4. Subtract:

$+18$	$+4$	-8	0	-12
-16	-4	-8	-5	6

5. In a certain game one couple made a score of 320, while another couple made a score of -40. By how many points were the latter couple defeated?

6. At noon the thermometer stood at $+10°$; at six o'clock it stood at $-8°$. How many degrees had the temperature fallen in the meantime?

7. The height of Mt. Everest is $+29{,}012$ feet; the greatest known depth of the ocean is $-31{,}614$ feet. Find the difference in elevation between these points.

8. King David died in 1015 B.C. (-1015), Queen Victoria in 1901 A.D. $(+1901)$. Find the number of years between these dates.

9. Find the value of $\dfrac{a+b}{c} - \dfrac{b+c}{d}$, if $a = 2$, $b = 3$, $c = 5$, and $d = 8$. Could we find the value of this expression if either c or d were 0? Why?

Using the values given in example 9 evaluate the following:

10. $3c(-4d) + \dfrac{(-b)(-d)}{a}$.

11. $\dfrac{a}{b+c+d} - \dfrac{b}{a} - \dfrac{c}{a} - \dfrac{d}{a}$.

12. $\dfrac{+d}{-b} - \dfrac{a}{c}$.

13. $\dfrac{abc}{10} - \dfrac{d}{b+c}$.

14. On eight examination questions John received the following deductions for errors: $-2, 0, -3, -1, 0, -6, 0, -5$. What was his final standing based on 100 credits?

15. The width of a rectangle is w feet. What is the length if its length l exceeds the width by x feet?

What is the perimeter of this rectangle when $w = 6$ feet and $x = 4$ feet?

16. Add:

$+5$	-10 dollars	17 feet
-10	-20 dollars	-8 feet
$+7$	$+30$ dollars	-12 feet

17. Subtract:

+ 75	− 15	− 18	− 16	+ 128	− 49	0	0
− 8	+ 28	− 31	− 14	− 128	+ 49	− 65	+ 65

18. Multiply:

+ 8	− 8	+ 8	− 8	− 6	+ 7	− 6	+ 12
+ 9	− 9	− 9	+ 9	− 5	− 5	+ 7	− 11

19. If a line 49 inches long is divided into three parts such that the middle part is twice the length of the smallest part, and the longest part is twice the length of the middle part, what is the length of each part?

20. What is the area (correct to the nearest tenth) * of a rectangle 7.8 inches long and 6.7 inches wide?

21. Which way do you count on the scale and at what do you arrive when you subtract from + 5 the following: + 6? + 7? + 8? + 10? − 2? − 3? − 4? − 6?

22. Solve the following and check:

(a) $3x + 8 = 20$.

(b) $4x - 5 = 31$.

(c) $\frac{1}{2}x + 6 = 18$.

(d) $3x + 4x + 5x = 72$.

Optional

Find the value of each of the following:

23. 3^4. *Hint:* 3^4 means $3 \cdot 3 \cdot 3 \cdot 3$.

24. $(-2)^2$; $(-3)^3$; $2(-4)^2$; $-3(-2)^4$.

25. $(-2)^4$; $(-4)^3$; $(-1)^3$.

26. $(-2)^5$; $(-2)^3(-3)^2$.

27. $(-4)^3(+2)^4$.

28. $(-1)^4(-3)^3(-2)^4$.

29. $(+3)^2(-4)^3(-5)^3$.

30. $(-2)^5$.

31. $(+3)(-2)(-4)$.

32. $(-2)^2(-4)$.

* Sir John Napier in 1600 was the first writer to use the decimal point as we use it to-day.

Cumulative Review

1. If $a = 3$ and $b = 4$, does $5a + 4b = 25$?

2. Does $7x + 5x - 6 = 30$ when $x = 3$?

3. The perimeter of a certain square is 48 inches. What is the length of each side of the square?

4. Find the value of x in the equation

$$6x + 5x + 2x - 3x = 20.$$

5. How do we evaluate algebraic expressions? Give an example.

Write formulas for

6. The velocity (V) of an object moving at the same uniform speed is the distance d divided by the time t.

7. The total number of seats (T) in a schoolroom is the product of the number (n) of seats in each row and the number of rows (r).

8. The physics law that the force (F) equals the product of the mass (m) and the velocity (v) divided by the time (t) in seconds.

9. The total cost (C) of (M) thousands of shingles at \$7 per thousand.

10. The total cost (C) of (M) thousands of cubic feet of gas at \$1.15 per thousand.

Give in words the meaning of each of the following:

11. (a) $2n$. (c) $9ab$. (e) $x^2 - z^2$.
(b) $5x = 15$. (d) x^2. (f) $4x - n$.

Solve each of the following equations for the letter:

12. (a) $x + 15 = 30$. (d) $5x - 9 = 51$.
(b) $x - 30 = 60$. (e) $\frac{5}{8}x = 40$.
(c) $3x + 8 = 44$. (f) $6.5x = 130$.

13. Is 15 the sum, product, or quotient of 5 and 3? of 8 and 7?

14. Solve:

(a) $\frac{x}{3} = 7$ (d) $\frac{x}{2} + 5 = 10$

(b) $\frac{x}{4} = 11$ (e) $\frac{x}{3} + 7 = 12$

(c) $\frac{x}{9} = 12$ (f) $\frac{x}{4} - 6 = 2$

15. A man travels k miles the first day and increases by 10 miles each day the distance traveled the preceding day.

(a) How far does he travel the second day? The third day? The fourth day?

(b) Write the formula for the distance d he will travel the nth day.

(c) Check by letting $k = 50$ and $n = 2$.

16. Express the following rule as a formula: The dividend (D) equals the product of the divisor (d) and the quotient (Q), plus the remainder (R).

17. In the following table for a graph determine the value of x corresponding to the value $y = 18$:

x	1	2	3	..	?	..
y	3	6	9	..	18	..

18. If $x = 3$, find the value of $x^2 - 7x - 9$.

CHAPTER IV

POLYNOMIALS. ADDITION AND SUBTRACTION

We have seen how algebra uses letters and negative numbers in various algebraic expressions to save time and make our work easier. We have solved simple problems with formulas and equations. To work more complicated examples we must master more complex expressions, such as those to be treated in the present chapter. To get a clear idea of these, we must learn their definitions.

30. Terms in Algebraic Expressions. — A *term* is that part of an algebraic expression contained between two consecutive + or − signs. Thus, in the expression $\frac{a+b}{3} + \frac{2}{xy} - 5$, $\frac{a+b}{3}$ is a term, $+\frac{2}{xy}$ is a term, and -5 is a term. Also see section 41.

The *sign of a term* is the sign which immediately precedes it.

A *positive term* is a term preceded by a + sign, expressed or implied.

A *negative term* is one preceded by a − sign.

A *monomial* contains only one term, as $6x$.

A *binomial* contains two terms, as $5x + 7y$.

A *trinomial* contains three terms, as $x + 6y + 2z$.

A *polynomial* is any expression of two or more terms. When the first term of a polynomial is positive, its sign is usually omitted.

31. Coefficients. — When we multiply two numbers to form a product, we call each the *coefficient* of the other. Thus in $2x$, 2 is the coefficient of x, and likewise x is the coefficient

of 2. In the expression $2ax$, $2a$ is the coefficient of x, 2 the coefficient of ax, etc. We use the word more commonly, however, for the numerical factor in cases like the above. If no numerical coefficient is written, the coefficient **1** is implied.

32. Exponents. — We have already seen that the expression ab means a multiplied by b If we multiply a by a, we do not write aa, but a^2 instead. In like manner $a \cdot a \cdot a$ is not written aaa, but a^3, and $aaaa$ is written a^4. These expressions are read "a square," "a cube," and "a fourth," respectively. The small figures ^{2}, ^{3}, and ^{4} in the illustrations are called *exponents*, and indicate how many a's are multiplied together in each case. This is another use of shorthand in algebra. If no exponent is written, it is understood to be 1. We may now state the definition.

An *exponent* is a small figure or letter written above and at the right of a quantity, to show how many of those quantities are multiplied together.

What does the expression x^4 mean when written at full length? State the value of 5^2, of 2^2, of 7^2, of 2^3, of 3^3.

33. Power. — A *power* of a number is the product of that number multiplied by itself one or more times. The number multiplied is called the *base*. The power is indicated by the exponent in each case.

Illustrations:

$3^2 = 3 \cdot 3 = 9.$	9, then, is the second power of the base 3.
$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$	16, then, is the fourth power of the base 2.
$(\frac{1}{2})^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$	$\frac{1}{8}$, then, is the third power of the base $\frac{1}{2}$.
$b^2 = b \cdot b.$	b^2, then, is the second power of the base b.

34. Similar Terms are terms which are alike in their literal (letter) parts. They may differ only in signs and numerical coefficients, for example, $2\,ab$, $-3\,ab$, $5\,ab$, $8\,ab$ are similar terms. So also are $3\,x^2y$, $-5\,x^2y$; and $4\,a^2bc$, $8\,a^2bc$.

35. Addition of Monomials. — Only similar terms can be added or subtracted. For others the operation must be *indicated;* it cannot be performed. Suppose we have to add the similar terms $6\,x^2y$, $10\,x^2y$, $-4\,x^2y$, and $-7\,x^2y$.

Rule. — (1) *Place the monomials in a vertical column.* (2) *Add the numerical coefficients and bring down the literal parts.*

$$\begin{array}{r} 6\,x^2y \\ 10\,x^2y \\ -4\,x^2y \\ \underline{-7\,x^2y} \\ 5\,x^2y \end{array}$$

If preferred, the terms may be arranged horizontally, writing the whole as a polynomial, and then *collecting* or combining the terms. Thus,

$$6\,x^2y + 10\,x^2y - 4\,x^2y - 7\,x^2y = 5\,x^2y.$$

Sum is the result of addition.

Exercises

Illustrative example:

Add and check with $x = 3$, $y = 2$.

$$\begin{array}{r} 12\,x^2y = 216 \\ 7\,x^2y = 126 \\ \underline{4\,x^2y = 72} \\ 23\,x^2y = 414 \end{array}$$

Add and check:

1.	2.	3.	4.	5.
7 inches	10 nickels	$+5\,n$	$-9\,c$	$-12\,x$
6 inches	8 nickels	$-8\,n$	$+14\,c$	$+7\,x$
6.	**7.**	**8.**	**9.**	**10.**
$-9\,x$	$+8\,x$	$-19\,x$	$-14\,a$	$+15\,z$
$-7\,x$	$-13\,x$	$+12\,x$	$+19\,a$	$-17\,z$

11.	12.	13.	14.	15.
$-4x$	$-15b$	$-20x$	$-30x$	$-25x$
$+7x$	$-13b$	$+40x$	$-10x$	$+15x$

16.	17.	18.	19.	20.
$+6x$	$-6x^2$	$-3xy$	$+7ab^2$	$-6xyz$
$-7x$	$-2x^2$	$+7xy$	$-5ab^2$	$-7xyz$
$+3x$	$+3x^2$	$-9xy$	$+ab^2$	$+3xyz$

21.	22.	23.	24.	25.
$3abc$	$-3x$	$-6bcd$	$6ax$	$-9xyz$
$8abc$	$5x$	$-12bcd$	$10ax$	$+14xyz$
$-5abc$	$-x$	$-15bcd$	$-4ax$	$-7xyz$
abc	$12x$	$-bcd$	$15ax$	$-3xyz$
$-12abc$	$9x$	$+8bcd$	$-3ax$	$+5xyz$

Collect (add):

26. $(+3x)+(-9x)$.

27. $(+11x)+(-7x)$.

28. $(-14x)+(+9x)$.

29. $(-15x)+(-5x)$.

30. $(+20x^2)+(-5x^2)$.

31. $3n+5n+7n-10n-4n+8n-3n+n$.

32. $-2b+5b-3b+4b-10b-2b-b+5b$.

33. $10x-2x+5x+3x-8x-12x-4x-x$.

34. $12x^2+8x^2-9x^2-5x^2-4x^2$.

35. $-15x^2+5x^2+20x^2-7x^2-5x^2$.

36. $20mn+3mn-15mn+2mn-10mn$.

37. $4a^2b+12a^2b-3a^2b-15a^2b+a^2b$.

38. $-10x^2y^2+16x^2y^2-5x^2y^2-12x^2y^2$.

39. $15x^2y-5x^2y+7x^2y-9x^2y-8x^2y$.

Optional

Add:

40.	41.	42.	43.	44.
$-\frac{1}{4}ab$	$+\frac{2}{3}xy$	$-\frac{1}{5}x^2$	$+3.5x^2y^2$	$-.45x^2$
$\frac{1}{2}ab$	$-\frac{1}{2}xy$	$-\frac{1}{4}x^2$	$+5.7x^2y^2$	$+4.5x^2$
$-ab$	$+\frac{3}{4}xy$	$+\frac{11}{20}x^2$	$-10.5x^2y^2$	$-2.25x^2$

45.	46.	47.	48.	49.	50.
$5.5abx$	$-5x^n$	$+9x^ny^m$	$2HKY$	$-3.5e^x$	$-\frac{3}{4}x^{a+1}$
$7.2abx$	$+7x^n$	$-12x^ny^m$	$-5HKY$	$+2.5e^x$	$-\frac{1}{2}x^{a+1}$
$3.3abx$	$+3x^n$	$-8x^ny^m$	$9HKY$	$-5.5e^x$	$+\frac{7}{8}x^{a+1}$

36. Addition of Dissimilar Terms. — When terms are not similar, we may *indicate* their sum or difference by the proper sign, although the result cannot be written as a single term. Thus, the sum of ax and by may be indicated as $ax + by$.

Exercises

Add:

1. $4x, +5y$ **4.** $7c, +5d$ **7.** $by, -dz$

2. $7x, -4y$ **5.** $12d, -3q$ **8.** $hk, +mn$

3. $9x, -13y$ **6.** $ax, +cy$ **9.** $ax, +by, -ez$

37. Addition of Polynomials. — For the addition of polynomials we have the following

Rule. — *To add two or more polynomials set them down so that similar terms fall in vertical columns, and add each column separately.*

Illustrative examples:

Check by letting $x = 2$.

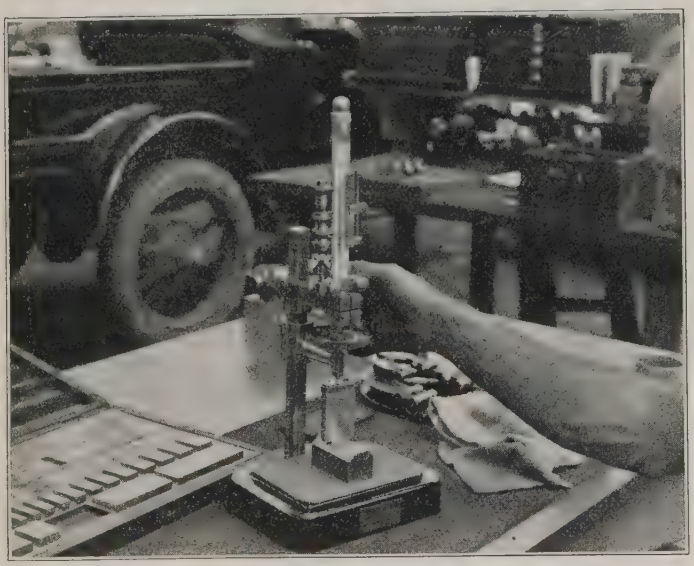
1. Add $3x - 2 = 6 - 2 = +4$

$4x + 8 = 8 + 8 = +16$

Sum $7x + 6 = 20$ $\quad 20$

2. Add $5x + 2y - 3z$, $4x - 6y + 5z$, and $-3x - 4y + 2z$ and check with $x = 2$, $y = 3$, and $z = 4$.

Solution:		*Check:*
	$5x + 2y - 3z =$	$10 + 6 - 12 = +4$
	$4x - 6y + 5z =$	$8 - 18 + 20 = +10$
	$-3x - 4y + 2z =$	$-6 - 12 + 8 = -10$
Sum	$6x - 8y + 4z =$	$12 - 24 + 16 = +4$



CHECKING

This standard gauge is used in the automobile industry to test the accuracy of the measurements.

38. Checks. — A check is a process of proving the correctness of an operation. To be valuable it must be reliable, and should, if possible, be simpler than the operation itself.

We may check addition by substituting for each letter some numerical value (other than 0). Thus, in the above example we may substitute 2 for x, 3 for y, and 4 for z.

Oral Exercises

Add the following:

1. $x + y$
   $\underline{x - y}$

2. $2x + y$
   $\underline{3x + y}$

3. $x + y + z$
   $\underline{2x + y + 3z}$

4. $a - 3b$
   $\underline{-a + 4b}$

5. (a) Add 3 to $x - 2$. (b) n and $2n - 1$. (c) $2x$ and $x - 3$.

6. To the sum of x and $x + 2$ add the sum of $2x$ and $x - 2$.

7. Collect terms in $3x + 2 + 6x - 2 + 4x + 1$.

8. Collect $\frac{1}{2}x + 3 + \frac{1}{4}x - 2$.

9. Collect similar terms in the expression:
$4x + 2y - 10 - 6y + 8x + 3y + 7 - 2x - 3$.

10. Add $x - y$
    $\underline{+ 2y + 3}$

11. Add $3m + n - 1$
    $n + 1$
    $\underline{4m - 2n}$

12. Add: $2x - 3y + 5$
    $x - y - 5$
    $\underline{-3x + 4y}$

13. Add $2x + 3y + 2$
    $3x + 2y + 1$
    $\underline{5x - 2y - 3}$

Exercises

Add and check, using suggested values for x, y, and z.

1. $5x + 6$ $x = 3$
   $\underline{3x - 8}$

2. $2x - 7$ $x = 2$
   $3x + 9$
   $5x + 2$
   $\underline{-4x - 1}$

3. $7x + 5y - 6$ $x = 3$
   $-2x - 6y + 7$ $y = 2$
   $-4x - 2y + 4$
   $\underline{2x + 7y + 2}$

4. $a - 6b + 7c$ $a = 3$
   $a - 4b + 2c$ $b = 2$
   $2a + 6b - 7c$ $c = 4$
   $\underline{3a + 4b + 2c}$

5.
$$\begin{array}{rl} 15x+17y+24z & x=2 \\ 12x+14y+15z & y=3 \\ 13x+12y+22z & z=4 \\ \underline{-12x-17y-24z} & \end{array}$$

6.
$$\begin{array}{rl} 5x+7y+3z+8 & x=2 \\ 2x-9y-4z-10 & y=3 \\ \underline{2x+6y-z-2} & z=4 \end{array}$$

7.
$$\begin{array}{rl} 6x^2+7y^2+8 & x=3 \\ 5x^2+3y^2+2 & y=2 \\ \underline{4x^2+4y^2+1} & \end{array}$$

8.
$$\begin{array}{rl} 2x^2+12x-15 & x=2 \\ 3x^2+14x+14 & \\ \underline{5x^2-25x+2} & \end{array}$$

9.
$$\begin{array}{rl} 5x^2+6x+7 & x=2 \\ 6x^2+7x+2 & \\ 3x^2-4x-5 & \\ \underline{4x^2-2x-4} & \end{array}$$

10.
$$\begin{array}{rl} 5x^3+4x^2+6x+7 & x=3 \\ 4x^3+5x^2+3x+4 & \\ \underline{6x^3+2x^2+2x+5} & \end{array}$$

11.
$$\begin{array}{rl} 6x^3-7x^2+8x-7 & x=2 \\ -4x^3+6x^2-7x+8 & \\ \underline{-2x^3+4x^2+4x-1} & \end{array}$$

12.
$$\begin{array}{rl} 5x^4+6x^3+3x^2-12 & x=2 \\ -4x^4+3x^3+2x^2-9 & \\ \underline{2x^4+4x^3+5x^2+21} & \end{array}$$

13.
$$\begin{array}{rl} 5.6x^2+3.5 & \\ -3.4x^2+6.5 & x=10 \\ \underline{-6.2x^2+5.4} & \end{array}$$

Arrange similar terms in columns, add, and check:

14. $3x+2y, 9x+y, 2x+5y.$ $x=2, y=3.$

15. $4x+7y, -6x-5y, 5x-2y.$ $x=2, y=4.$

16. $5x+6y-z, -2x-4y+9z.$ $x=2,\ y=3,\ z=4.$

17. $5x^2+7x-12, x^2-9x+4.$ $x=2.$

18. $3x^2-9x+15, -2x^2+6x-10.$ $x=2.$

19. $4x^2+2xy+y^2, x^2-5xy-7y^2.$ $x=3, y=2.$

20. $2x^2-3xy-4y^2,\ x^2-2xy+y^2.$ $x=3,\ y=2.$

21. $9(x+y)+6z,\ 7(x+y)-4z.$ $(x+y)=3,\ z=1.$

Select your own numbers for checking.

22. $6m-3n+4x+10;$ $-8m+5n+3x-2;$ $2m-7n-18;$ $8n+5x-12.$

23. $5a - 3b + c$, $4a + 6b - 3c$, and $6a - 9b + 4c$.

24. $6x + 2y - 8z$, $3x - 7y + 2z$, and $8y - 9z + 6x$.

25. $3h + 4k - 5m$, $6k + 5h + 3m$, and $7m - 4h + 9k$.

26. $3x - 8y + 2k + l$; $3x - 10y - 18k + 4l$; $-5x + 3y - 10k + 8l$; $x + y + k + l$.

27. $3h^2 + 4hk - 5k^2$; $h^2 - k^2$; $8h^2 - 2hk + 11k^2$; $-4h^2 + 12hk - 17k^2$.

28. $5ab - 3ac + 6bc$, $5ac - 6ab + 3bc$, and $8ab - 7bc + 4ac$.

29. $x^4 - 10x^3 + x^2 + 10$; $3x^3 - 6x^2 + 8x - 2$; $2x^4 + 10x^2 + 8x - 3$.

30. $y^3 - 4y^2 - 3y$; $12y^2 - 2y + 5$; $8y^3 - 11y + 7$; $-8y^3 - 10y^2 + 16$.

31. $12x^3 - 6 + 4x - 13x^2$ and $+5 - 8x + 18x^2 - 9x^3$.

32. $5x - 7x^3 + 6$, $7x^2 - 4x + 10x^3$, and $-10 - 3x^3 + 8x - 7x^2$.

33. $6x^2 - 4y^2$, $-4xy + 7y^2 - 4x^2$, $5x^2 - 7xy$, and $7x^2 + 5xy - 9y^2$.

34. $a^3 + 3a^2b + 3ab^2 + b^3$, $a^3 - 3a^2b + 3ab^2 - b^3$, and $4a^3 - 6ab^2$.

Collect (try to add-in-your-head):

35. $3a + 4b + 2c + 5a - 6b - 3c - 7a - 8b - 4c$.

36. $-5y + 2x - 1 + 3x + 3y - 2 - 2x + 2 - x - 3 - 3y + 1$.

37. $6x - 4y + 2z - 8x - 6y - 3z + 9x - 10y - 7z$.

38. $7a - 9b + 3c + 3a - 11b + 7c - 10a + 10b - 10c$.

39. $x + 4y - 5z + 7y - 2z + 3x - 10z + 6y + 9x$.

40. $5x + 3 - 2y + 7 - 2x + 5y - 10x + 4 - 5y$.

41. $6x^2 + 3xy + y^2 - 9xy + 7y^2 - 8x^2 + 4xy + 2x^2 - y^2$.

42. $7a^2 + 2ab + b^2 - 4b^2 - 4ab - 5a^2 + 6ab + b^2$.

43. $9x^2 + 7y^2 - 16z^2 + 8y^2 + 7z^2 - 8x^2 - 15y^2 - x^2$.

Optional

44. $5.6\,x + 6.5\,y$, $3.4\,x + 5.2\,y$, and $6.8\,x + 3.5\,y$.

45. $7.5\,x + 8.5\,y$, $2.5\,x + 1.5\,y$, and $4.5\,x - 7.5\,y$.

46. $4.5\,a - 3.6\,b$; $4.2\,a + 3.6\,b$; $-3.1\,a + 2\,b$; $a - 5.8\,b$.

47. $3\frac{1}{2} - 2\frac{1}{4} + 5\frac{1}{2} - 3\frac{3}{4} - 4 + 2$.

48. $\frac{1}{3}\,x + \frac{1}{2}\,y$; $\frac{1}{2}\,x - \frac{1}{3}\,y$; $x - y$.

49. $\frac{1}{2}\,N - \frac{1}{4}\,M$; $3\frac{1}{2}\,N + \frac{1}{2}\,M$; $-\frac{1}{2}\,N + \frac{1}{4}\,M$.

50. $2\frac{1}{3}\,H + \frac{3}{5}\,K$; $\frac{2}{3}\,H + \frac{1}{4}\,K$; $3\,H - 2\,K$.

Timed Practice Test. 5 Minutes

Copy and add:

1. $-12\,xy$ / $+15\,xy$

2. $6\,bc$ / $2\,bc$

3. $15\,cd^2$ / $-23\,cd^2$

4. $-7\,c^2d$ / $9\,c^2d$

5. $-7\,abc$ / $+7\,abc$

6. $49\,y^2$ / $15\,y^2$

7. $8\,a - 6\,b$ / $3\,a - 5\,b$

8. $3\,x - 4\,y$ / $4\,x - 3\,y$

9. $-4\,a^2 - 7\,b^2$ / $-6\,a^2 - 7\,b^2$

10. $-8\,y^2 - 3\,yz$ / $-3\,y^2 - 4\,yz$

11. $4\,a^2 - 5\,y^2$ / $4\,a^2 - 7\,y^2$

12. $-3\,a - 4\,b$ / $7\,a + 7\,b$

13. $-25\,x^2 - 9$ / $13\,x^2 - 8$

14. $5\,a^2 + 3\,ab - 6\,b^2$ / $-7\,a^2 + 4\,ab - 3\,b^2$

15. $-13\,ab - 17$ / $35\,ab - 18$

16. $6\,x^3 - 4\,x^2 + 7$ / $-9\,x^3 + 5\,x^2 - 7$

39. Subtraction of Monomials. — Similar monomials may be subtracted by the following

Rule. — *Subtract the coefficient of the subtrahend from that of the minuend, and annex the like part of the terms.*

Illustrative examples:

10 nickels	8 dimes	$2x$	$-7x$
5 nickels	6 dimes	$3x$	$+2x$
5 nickels	2 dimes	$-1x$	$-9x$

$+20x$	$+20x$	$-20x$	$17x^2$
$-2x$	$+2x$	$-5x$	$-3x^2$
$+22x$	$+18x$	$-15x$	$20x^2$

NOTE. Refer to rule on page 77.

If the monomials are not similar, the subtraction of one number from another can only be indicated. For example, to subtract y from x we write $x - y$.

Exercises

Subtract the lower expression from the upper and check by adding the remainder to the subtrahend, giving the minuend.

1.	2.	3.	4.	5.	6.	7.	8.
$17x$	$-10x$	$-2x$	$-7x$	$+11x$	$10x$	$-5x$	$-9x$
$-5x$	$+5x$	$-8x$	$+9x$	$-3x$	$-10x$	$+5x$	$-3x$

9.	10.	11.	12.	13.	14.
$-6y$	$7x$	$+11y^2$	$-12xy$	$+9xy$	$-15xy$
$+9y$	$-3x$	$-4y^2$	$+3xy$	$-7xy$	$-8xy$

15.	16.	17.	18.	19.	20.
$17xy$	$-6x$	$40xy$	$-75a$	$-36xyz$	$-9xyz$
$-5xy$	$+3x$	$-20xy$	$+25a$	$+14xyz$	$-9xyz$

21.	22.	23.	24.	25.	26.
$-3a^2b^2$	$7.5x$	$8.4x$	$6.2x$	$-15x^2$	$25x^3$
$+2a^2b^2$	$3.5x$	$3.5x$	$4.7x$	$-10x^2$	$-25x^3$

27.	28.	29.	30.	31.
$-30x^4$	$40x^4$	$-6a^2x$	$15x^2$	$100x^2$
$-30x^4$	$-20x^4$	$-a^2x$	$-30x^2$	$-50x^2$

32.	33.	34.	35.	36.
$-100x^2$	$-100x^2$	$70x^2y$	$-35x^2yz$	$-10x^3y$
$-50x^2$	$+50x^2$	$-20x^2y$	$-21x^2yz$	$-7x^3y$

37.	38.	39.
$7(x+y)$	$14\sqrt{x}$	$+12.4$
$5(x+y)$	$6\sqrt{x}$	$+6.3$

By the use of the minus sign, subtract:

40. a from b.
41. $7x$ from $4y$.
42. $9x$ from $7z$.
43. $5y$ from $6x$.
44. $6a$ from $5b$.
45. $3x^2$ from $5y^2$.
46. $4y^2$ from $3z^2$.
47. $3z^2$ from $4y^2$.
48. $7x$ from 0.
49. $9x^2y$ from $12ab$.

50. What must be added to $5x$ to produce $-7x$?

51. What single term is the same as $8x - 5x + 4x$?

52. Subtract $4x$ from 0.

53. (a) Take $3x$ from $4x$.
(b) Subtract $4x$ from $3x$.
(c) Which is the correct form for (a), $3x - 4x$ or $4x - 3x$?

54. By first subtracting $12xy$ from $14xy$ before substituting the values of x and y, find the value of $14xy - 12xy$, when $x = 2$ and $y = 3$.

55. If $x = 6$, what is the value of (a) $17x - 15x$?
(b) $15x - 17x$?

56. If a man decreases his debts, is he then richer or poorer?

57. If a boy loses 5 dollars 25 cents of his employer's money and the boy had no money of his own, what then will represent his money condition? 5 dollars 25 cents subtracted from $0 = ?$

True-False Exercises

Copy the numbers of the following examples, placing a t after each of those that you know are correct, and an f after each one that you believe is incorrect:

1. The sum of $7 - 13 + 4 - 10$ is -11.

2. The sum of $3x^2y$ and $-7x^2y$ is $-4x^2y$.

3. The sum of $8a^3b + 6a^3b - 9a^3b$ is $5a^3b$.

4. The sum of $9a + 10b^2 - 4a - 7b^2$ is $5a$.

5. The exponent of y in $4y^5$ is 4.

6. The coefficient of x^2y in $-9x^2y$ is -9.

7. The coefficient of b in $6a^2b$ is $6a^2$.

8. The numerical coefficient of $-xyz$ is -1.

9. The exponent of a in $12ab^2$ is zero.

10. $9a - 7b^2$ taken from $7a + 2b^2 = -2a + 9b^2$.

11. $x + 2y = 16$ if $x = 4$ and $y = 6$.

12. $-3xy = -24$ if $x = 2$ and $y = 4$.

13. The exponent of b in $11a^2b$ is 1.

14. The coefficient of x in $7x$ is 7.

15. The value of $9a^2 - 30b$ is 7 when $a = 3$ and $b = 2$.

16. If we add $6(a + b)$ and $-9(a + b)$, we obtain $15(a + b)$.

17. If we subtract $6(a + b)$ from $-7(a + b)$, we obtain $-13(a + b)$.

18. $x + y + z + 3x + 4y - z - 4x - 5y + z = z$.

19. $\frac{1}{3}x - \frac{1}{4}y + \frac{1}{2}z - \frac{1}{2}x + \frac{1}{4}y - \frac{1}{2}z = -\frac{1}{6}x$.

20. $.5H - .7K + .3H - .3K - .7H + .8K = H + .2K$.

40. Subtraction of Polynomials.

Rule. — *To subtract polynomials having similar terms,* (1) *place the subtrahend under the minuend so that similar terms fall in vertical columns.*

(2) *Change the signs of the terms of the subtrahend, mentally* (see section 27), *and add the coefficients.*

Illustrative example:

From $3x^2 + 7xy - 4y^2 - 1$ subtract $-7x^2 + 4xy - 3y^2 + 2$ and check, substituting 2 for x and 3 for y.

Solution:

$$\begin{array}{rr} 3x^2 + 7xy - 4y^2 - 1 & = 17 \\ -7x^2 + 4xy - 3y^2 + 2 & = -29 \\ \hline 10x^2 + 3xy - y^2 - 3 & = 46 \quad 46 \end{array}$$

Check by addition:

$$\begin{array}{ll} -7x^2 + 4xy - 3y^2 + 2 & \text{subtrahend} \\ 10x^2 + 3xy - y^2 - 3 & \text{remainder} \\ \hline 3x^2 + 7xy - 4y^2 - 1 & \text{minuend} \end{array}$$

Oral Exercises

What number must be added to the lower number to give the upper number? Subtract:

1. $\begin{array}{r} a + b \\ \underline{a - b} \end{array}$

2. $\begin{array}{r} x + 2y \\ \underline{x - 2y} \end{array}$

3. $\begin{array}{r} 3x - 5y \\ \underline{2x + 5y} \end{array}$

4. $\begin{array}{r} 2x + 3y - 5z \\ \underline{x + 4y - 4z} \end{array}$

5. $\begin{array}{r} 4x^2 - 3y^2 \\ \underline{2x^2 - 6y^2} \end{array}$

6. $\begin{array}{r} \frac{1}{2}x + \frac{1}{3}y \\ \underline{\frac{1}{2}x - \frac{2}{3}y} \end{array}$

7. $\begin{array}{r} 6 \text{ bu.} + 12 \text{ qt.} \\ \underline{5 \text{ bu.} - 8 \text{ qt.}} \end{array}$

8. $\begin{array}{r} 8 \text{ ft.} + 10 \text{ in.} \\ \underline{3 \text{ ft.} + 17 \text{ in.}} \end{array}$

9. 8 degrees + 14 minutes + 5 seconds
12 degrees + 14 minutes − 5 seconds

Exercises

Subtract:

1. $6x - 9y + 10z$
$4x + 5y - 3z$

2. $12x + 15y - 10z$
$5x + 8y - 12z$

3. $3x + 6y - 9z$
$2x - 3y + 3z$

4. $10x^2 - 7x - 5$
$8x^2 - 4x + 3$

5. $5x^2 + 11x - 12$
$-2x^2 - 3x - 7$

6. $4x^2 - 6xy + y^2$
$3x^2 + 6y^2$

Subtract the first quantity from the second. Check:

7. $x + 4, x + 5.$
8. $x - 6, x - 7.$
9. $2x - 3, 4x + 6.$
10. $5x + 4, -7 + 6x.$
11. $3a - 6, 6 - 3a.$
12. $7a + 5, 5a + 7.$
13. $0, x + 6.$
14. $3x + 5, 0.$
15. $x + 2y + 3z, 3x - 2y - z.$
16. $6h + 5k - 4m, 4h + 6k - 4m.$

17. From $15x - 6y + 8z$ subtract $6x - 7y + 10z$.
18. From $9x^2 - 5x + 3$ subtract $5x^2 - 6x + 4$.
19. From $8x^3 + 7x^2 - 6$ subtract $3x^3 - 5x^2 - 8$.
20. Subtract $7x^2 - 3xy + 8y^2$ from $10x^2 - 5xy + 6y^2$.
21. Subtract $10x^2 - 5x + 10$ from $15 + 6x - 7x^2$.
22. From $7x + 12 - 3x^2$ subtract $5 - 8x - 4x^2$.
23. From $5y - 10x + 3z$ subtract $5x + 6y - 4z$.
24. From $7x - 8x^2 - 4$ subtract $6 + 9x^2 - 2x$.
25. From $9x^2 - 4x + 3x^3$ subtract $3x^2 - 5x^3 + 6x$.
26. From $10x + 3x^2 - 6$ subtract $7 - 4x^2 + 12x$.

27. From $8x^2 - x + 7$ subtract $9 + 2x + 5x^2$.

28. Subtract $x^2 - 3y^2$ from $3x^2 + 7y^2$.

29. Subtract $5x^2 - 2y^2$ from 0.

30. Subtract $x - y$ from $-x - 11y$.

31. Subtract $3x+4y-7z+11$ from $2x-3y-9z-12$.

32. From 0 subtract $x + 3y - 6z$.

33. Subtract $-4x + y - 3z$ from 0.

34. From $-6x + 2y + 4z$ subtract $-5x + 2y - 4z$.

35. Subtract $x - y + z$ from $x + y + z$.

What must be added to the first expression to produce the second?

36. $a + b + c$, $2a + 3b + 4c$.

37. $2a + 3b - c$, $3a - 2b - c$.

38. $3a + 4b - 5c$, 0.

39. $4x + 5y$, $3a + 5k$.

What expression must be subtracted from the first to produce the second?

40. $3x + 2y$, $x + y$.

41. $5x - 4y$, $4y - 5x$.

42. $3a + 2b + c$, $3a - 2b + c$.

43. $5x^2 + 6y^2$, $7x^2 + 6y^2$.

44. $7x + 5y + 4z$, 0.

45. $3x + 4y + 3z$, $3x + 4y$.

46. From $x^4 + 6x^2 - 8x + 3$ take $3x^3 + 6x^2 + 5x - 1$.

47. From $12 + 4x - 8x^2$ take $17x^2 - 4x + 2$.

48. From -12 subtract $x^4 + 6x^2 + 16$.

49. From 0 subtract $-x^3 + 8x^2 + x - 3$.

50. From the sum of $2x^3 - 6x^2 + 8x - 1$ and $5x^3 + 2x^2 - 3x + 4$ subtract $8x^3 + 3x^2 - x + 2$.

Optional

51. From the sum of $5x^2 + 7x + 8$ and $6x^2 + 9x - 7$ subtract $5x + 3x^2 - 11$.

52. From the sum of $3x$ and $4y$ subtract their difference.

53. From the sum of $8x + 9x^2 + 3x^3 - 4$ and $7x^3 - 8x^2 + 5x - 12$, subtract $x + 2x^2 - 6x^3 + 6$.

54. From $6x^4 - 3x^3 + 5x^2 - x + 8$ subtract the sum of $2x^3 + 8x^2 - x + 3$ and $3x^4 + x^3 + 8x - 1$.

55. From $8x^3 - 4x + 3x^2 - 6$ subtract the sum of $7x^2 - 3x^3 + 5 - 2x$ and $9x - 4 + 2x^3 - 5x^2$.

56. From $5x^4 - 3x + 5x^2$ subtract the sum of $8x^2 - 7x + 9x^4$ and $3x + 5x^2 - 8x^4$.

57. From the sum of $2y^3 + 7y^2z + 4yz^2 - z^3$ and $y^3 - 6y^2z + 7yz^2 + z^3$ subtract the sum of $2y^2z - 6yz^2 + 4z^3$ and $3y^3 + 4y^2z + 6z^3$.

58. What must be subtracted from $3m^3 + 4m^2 - 8m + 3$ to leave $2m^3 + 7m^2 + m - 5$?

59. What must be added to x to produce y?

60. What must be added to $4x^4 - 3x^3 + x^2 - 10$ to produce $4x^4 + 6x^3 + 2x^2 - 8x + 3$?

61. By how much does the sum of $2n^2 - 3n + 1$ and $n^3 + 4n - 8$ exceed the sum of $4n^3 - 6n^2 + 3$ and $n^3 + 5n^2 - 7n$?

62. From $R^2 + 2RD + D^2$ take $4R^2 - 6RD$.

63. From $1.5r - s + 2.8t$ take $.8t - 2r - s$.

64. From $\frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{4}$ take $\frac{3}{2}x^2 - \frac{1}{2}x + \frac{3}{4}$.

65. From $12v' - 5 - 3v''$ take $-4v'' + 8 - 2v'$.

Hint: v' is read "v prime"; v'' is read "v double prime."

Time-Practice Test. 6 Minutes

Copy and subtract the following:

1. $6\,xy$ / $9\,xy$
2. $-7\,abc$ / $4\,abc$
3. $8\,y^2$ / $3\,y^2$
4. $16\,ab^2$ / $-4\,ab^2$
5. $-9\,x^2y^2$ / $9\,x^2y^2$
6. $36\,a^3b$ / $-15\,a^3b$
7. $-35\,x^2y$ / $-21\,x^2y$
8. $56\,a$ / $-17\,a$
9. $-17\,xy^3$ / $-19\,xy^3$
10. $3\,x - y$ / $4\,x - y$
11. $23\,x^2 - 3\,xy$ / $2\,x^2 - 8\,xy$
12. $7\,a^2 - 5\,a$ / $-5\,a^2 - 3\,a$
13. $-27\,by - 9$ / $17\,by - 11$
14. $5\,xy - 9$ / $3\,xy - 9$
15. $-8 - 3\,xy$ / $-8 - 4\,xy$
16. $62\,x^2y - 21\,x$ / $50\,x^2y - 30\,x$
17. $12\,a^3b - b$ / $-18\,a^3b - 7\,b$
18. $4\,x^3 + 2\,x^2 - 3\,x - 6$ / $5\,x^3 - 3\,x^2 - 3\,x + 6$
19. $5\,x^3 - 17\,x + 8$ / $-6\,x^3 + 19\,x - 9$
20. $7\,x^3 - 9\,x - 17$ / $-8\,x^3 + 11\,x - 19$

41. The Parenthesis. — We already know what a term is. The parenthesis,* ( ), is used to collect two or more terms into one. That is to say, an expression within a

* The term *parenthesis* as used here signifies the two upright curves taken collectively, not separately.

parenthesis is treated as a single quantity no matter how many terms it may contain. The entire parenthesis then becomes a single term or part of a term, as

$$3\,x(x-y)+4\,y(x^2+6\,xy+y^2-1).$$

Here we have two terms, namely, $3\,x(x-y)$ and $4\,y(x^2+6\,xy+y^2-1)$.

The *brackets*, [ ], *braces*, { }, and *vinculum*, $\overline{}$, may be considered as other forms of the parenthesis. They have the same meaning as the parenthesis, and the rules here stated concerning the use of the parenthesis apply equally to them.

In adding two or more algebraic expressions inclosed in parentheses, as $(x^2+6\,x-3)+(2\,x^2-x+6)$, we may simply omit the parentheses and combine similar terms, since we do not change the signs in addition. Thus we obtain $x^2+6\,x-3+2\,x^2-x+6=3\,x^2+5\,x+3$.

42. The Parenthesis in Subtraction. — The use of the parenthesis in subtraction is important. It furnishes a convenient method of indicating the subtraction of a polynomial. For example, let it be required to find the value of

$$(3\,x^3+x^2-x+1)-(4\,x^3+7\,x^2+x-6).$$

We may solve this example by the usual method of subtraction, namely:

$$\begin{array}{r} 3\,x^3+\ x^2-\ x+1 \\ 4\,x^3+7\,x^2+\ x-6 \\ \hline -\,x^3-6\,x^2-2\,x+7 \end{array}$$

Or we may proceed as follows: recalling that the expression within the second parenthesis is the subtrahend, and that we change the sign of the subtrahend in subtraction, we may omit the parentheses, changing the sign of all terms now inclosed in the second one, and then collect, or add terms. Thus,

$$(3x^3 + x^2 - x + 1) - (4x^3 + 7x^2 + x - 6)$$
$$= 3x^3 + x^2 - x + 1 - 4x^3 - 7x^2 - x + 6$$
$$= -x^3 - 6x^2 - 2x + 7.$$

From the above examples we deduce the following

Rule. — *A parenthesis preceded by a + sign may be omitted without any change of sign; a parenthesis preceded by a − sign may be omitted, provided the sign of every term inclosed within it is changed.*

In applying the above rule we should note that the sign which precedes a parenthesis is regarded as belonging to the parenthesis, and disappears when the parenthesis is removed. Thus, in the illustration above, when the parenthesis inclosing $4x^3 + 7x^2 + x - 6$ is removed its sign (−) disappears also. The sign of $4x^3$ is understood to be +. Now, when the parenthesis is removed, and the signs of all terms inclosed are changed, $+4x^3$ becomes $-4x^3$. In like manner, the signs of the other terms are changed.

Exercises

Remove parentheses, and combine terms when possible:

1. $6 + (2x - 3)$.
2. $5a + (b + 2a)$.
3. $10x - (2x - 5)$.
4. $5a - (3c - 2a)$.
5. $7b + (3b + 2c)$.
6. $2x - (3x + y - 2)$.
7. $5y - (y + 1)$.
8. $y + (1 - 3y)$.
9. $6y - (2 + 4y)$.
10. $y - (-y - 2)$.
11. $2x + (3x - y)$.
12. $3x + y - (4x - 2y)$.
13. $15 - (-8.3)$.
14. $5.7 - (-6.3) - 4.5$.
15. $.3 - (3x - .5)$.
16. $5.2 - (y - .8)$.
17. $3.5 - (.3 - 5x)$.
18. $9 - (-y - 2.5)$.
19. $(3x^2 - 4x + 1) - (x^2 + 2x - 3)$.

20. $(x^2 - xy - y^2) - (-x^2 + 2y^2)$.

21. $x^2 - [x^2 - x - 1 + 1]$.

22. $a^2 - \{a^2 - b^2 - b^2 + a^2\}$.

43. Two or More Parentheses. — If two or more parentheses are used, one within another, it is best to remove them one at a time, beginning with the innermost.

Illustrative example:

Simplify: $x^2 - [2x^2 + x - (3x - 1) - x^2] + 1$.

Solution:
$$\begin{aligned} & x^2 - [2x^2 + x - (3x - 1) - x^2] + 1 \\ &= x^2 - [2x^2 + x - 3x + 1 - x^2] + 1 \\ &= x^2 - 2x^2 - x + 3x - 1 + x^2 + 1 \\ &= 2x. \end{aligned}$$

Exercises

Simplify:

1. $6 - [8 - (5 - 12)] - (20 - 30)$.

2. $20 - (10 - 15) - [(5 - 2) - 7]$.

3. $2a - [3a - (2a - b) + 3b] - a + b$.

4. $5x - [6x + (3x - 1)]$.

5. $(2D - \{3 - D\} + 1)$.

6. $3y + [8y - \{3y - 2\}]$.

7. $x + [4x - (5x + 3y)] + (4y - 3x)$.

8. $2n - \{3n - [n + 2] + 4n\} + 1$.

9. $9x - [6 - (6x - 7)] - 4$.

10. $-\{6x - (4x + 1) + 3x - 1\} - x$.

Optional

11. $y^2 + \{-y^2 + 3y - (2y^2 - y + 1) - y^2\} + 1$.

12. $6m - [6m^2 + (3m + 1) - m] - (m^2 + 1)$.

13. $2 - [-3 + \{6 - \overline{2 + 1}\} - 1]$.

14. $2x^2 - (x + [3x^2 - 1] + x^2 - 1) - 1.$

15. $(x^3 + 2x^2 - x + 1) - (3x^2 - 6x + 3) - (4x^3 + 6x^2 + x - 2).$

16. $s - \frac{1}{2}at + \{2\frac{1}{2}at - (s - at)\} - \frac{1}{2}at.$

17. Find the value of x in $2x - (x - 2) = 8.$

18. Solve $5x - [8x - (2x + 7) - 1] = 0.$

44. Insertion within Parenthesis. — We often need to inclose two or more terms within a parenthesis. *We may inclose any number of terms within a parenthesis preceded by a + sign without change; but if we inclose them within a parenthesis preceded by a − sign, the sign of every term so inclosed must be changed,* otherwise when the parenthesis is removed we shall not obtain the original expression.

Illustrative example:

Inclose within a parenthesis preceded by a − sign the last three terms of the expression $x^2 + (y^2 - y + 3.)$

We obtain $x^2 - (-y^2 + y - 3).$

If the parenthesis be now removed according to the rule for subtraction on page 107, we obtain $x^2 + y^2 - y + 3$, the given expression.

Exercises

Inclose the last three terms in a parenthesis preceded by a minus sign:

1. $5x(-y + 6z - w.)$

2. $2a(-b - c + d.$

3. $8x(+5a + 6b - 2c.$

4. $x^2(-y^2 - z^2 + 2yz.)$

5. $4y^2(-x^2 + 2x - 1.)$

6. $x^2 + y^2 - 2y - 1.)$

7. Write $a^2 - b^2 - c^2$ so that the last two terms are in a parenthesis preceded by a plus sign.

Inclose the first three terms within a parenthesis preceded by a + sign, and the last three terms within a parenthesis preceded by a − sign:

8. $(x^2 - 6x + 9) - y^2 + 4y - 4.$

9. $(6m^2 - 8mn + 10) + 6n^2 - 8n - 3.$

Inclose every term containing x within one parenthesis preceded by a + sign, and every term containing y within another parenthesis preceded by a − sign.

10. $x^2 + 4y - 3x + 6y^2 - 2xz + 5yz.$

11. $3x + ay - ax + by^2 - y - 6bx.$

By the use of parentheses show that

12. $2x + 3y$ is to be subtracted from $9x - 4y$.

13. $3x - 4y - 5z$ is to be subtracted from $4x - 5y + 2z$.

14. $x^2 - 2xy + y^2$ is to be subtracted from z^2.

15. The sum of a and b is subtracted from their difference.

Review

If $a = 5$, $b = 2$, $c = -1$, $d = -4$, find the value of

1. $3a^2 - ab + c^2$.

2. $4a + 5b^2 + 8c$.

3. $\frac{a}{b} + \frac{b}{c} - \frac{c}{a} + d$.

4. $\frac{ab^2 - bc^2 + cd^2}{abcd}$.

5. $(a - b)^2 + cd^2$.

6. $\frac{a + b}{c} - \frac{c + d}{a}$.

7. What is the area of a triangle whose base is b and whose altitude is a? Find the area when $b = 8$ and $a = \frac{1}{2}$.

8. If the area of a rectangle is denoted by $A = lw$, substitute 6.4 for l and 3.8 for w, and find the value of A.

9. If the area of a circle is denoted by $A = \pi R^2$, find the area of a circle whose radius $(R) = 5$, and $\pi = 3.1416$.

10. Multiply each of the following:

$+6$	-11	$+11$	$+15$	-14
-7	-13	-11	-20	-100

11. What is the value of $x^5 \cdot x^3$ when $x = 2$?

12. By how much is $-12 + 6x - x^2$ greater than $x^3 - 4x^2 + 32$?

13. If
$$A = x^3 - 3x^2 + 1$$
$$B = 2x^2 - 5x - 3$$
$$C = 3x^3 + x^2 + 3x$$
find the value of $A + B - C$, $A + B + C$, $A - B + C$.

14. By how much does $x^3 + x^2 + x + 1$ exceed $2x^2 + 3x + 2$?

15. What is the number next larger than 4? than y? than $y + 1$? than $y - 1$?

16. (a) Express in briefest form:
$$9m^2 + 4m^2 - 10m^2 + 4m^2.$$
(b) Express the sum of x, y, z, and w.

17. By the use of exponents express in the shortest way:
$$7xxxxyyyzzzzz.$$

18. If a man drives an automobile at the rate of 35 miles per hour, how far will he drive in 4 hours? in h hours?

19. Write the sum of 15 and x; the difference; the product.

20. Express in algebraic symbols: six times x squared less five times y cubed.

21. Define: A term. An exponent. A power of a number.

22. A village newspaper printed 1400 copies. The number sold by newsboys was twice the number sent out by mail, and the number delivered by carriers to subscribers in the village was twice the number sold by newsboys. If the entire edition was disposed of, how many were mailed?

23. Solve the following equations:

(a) $x + 9 = 19$.
(b) $x + 5 = 40$.
(c) $x - 10 = 50$.
(d) $x - 50 = 150$.
(e) $5x + 9x + 7x = 84$.
(f) $19x + 12x - 5x = 78$.
(g) $3x + 8 = 29$.
(h) $5x - 10 = 50$.

24. Solve for x:

(a) $6x + 3x = 27$.
(b) $8x + 9x = 102$.
(c) $2x + 16x = 72$.
(d) $9x + 2x - 3x = 24$.
(e) $12x + 4x - 6x = 80$.
(f) $15x + 6x - 2x = 95$.

CHAPTER V

MULTIPLICATION AND DIVISION

After mastering the processes of addition and subtraction we must learn to multiply and divide algebraic expressions. In this chapter we shall see how to perform these operations, and how to apply them to the solution of equations.

45. Exponents in Multiplication. — The multiplication of similar monomials is made simple by the way the exponents are treated. The method of treating exponents in multiplication may be illustrated by the following

Illustrative example:

Multiply x^3 by x^4.

Solution:

$$x^3 = x \cdot x \cdot x$$
$$x^4 = x \cdot x \cdot x \cdot x$$

Therefore,

$$x^3 \cdot x^4 = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$$
$$= x^7.$$

How is the exponent 7 in the product related to the exponents 3 and 4 in the numbers multiplied? Why is this true?

Similarly, $x^2 \cdot x^3 = x \cdot x \cdot x \cdot x \cdot x = x^5$

and $2^3 \cdot 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8$.

Can you formulate a rule in words?

In symbols we may write $a^m \cdot a^n = a^{m+n}$. Then, $2^3 \cdot 2^2 = ?$ $3^2 \cdot 3^4 = ?$ $2 \cdot 2^5 = ?$

The law of signs in multiplication has already been explained in section 28.

Illustrative examples:

$+ 3a^2$	$- 6a^2x^3$	$3x^{n+1}$	ax^4
$- 4a^3$	$- 5ax^8$	$5x^{n+3}$	$- bx^5$
$- 12a^5$	$+ 30a^3x^{11}$	$15x^{2n+4}$	$- abx^9$

Product is the result of multiplication. How are coefficients in products obtained? How are exponents obtained?

*Rule.** — *To multiply two monomials,* (1) *multiply their coefficients to obtain the coefficient of the product.*

(2) *Add the exponents of like letters in each expression to obtain the exponents of those letters in the product.*

Exercises

Perform the following multiplications:

1. $x^3 \cdot x^5$.
2. $x^2 \cdot x^2$.
3. $x^3 \cdot x^3$.
4. $x^5 \cdot x^2$.
5. $x^6 \cdot x^4$.
6. $x^5 \cdot x^4$.
7. $x^4 \cdot x^2$.
8. $4x^2 \cdot x^3$.
9. $2x^4 \cdot (-4x)$.
10. $-3x^2 \cdot 5x^6$.
11. $(-6x)(-7x)$.
12. $(4x^3)(-3x^3)$.
13. $(5x^5)(6x^2)$.
14. $(-4x^4)(-3x^3)$.
15. $(-6x^2)^2$.
    $[(-6x^2) \cdot (-6x^2).]$
16. $(-3x)^2$.
17. $(-3x)(-3x)(-3x)$.
18. $(3x)(4y)(5xy)$.
19. $(ax)(5ax)$.
20. $-6ab$
    $-2a$
21. $-6ab^2$
    $-4ab^2$
22. $(+3a^3)(4a^2x)$.
23. $(4xy^2)^2$.

* Michael Stifel (1553) was the first to use the word "exponent," and to give us the rule which we now use for multiplying powers of the same base.

24. $3xy$ / $-\frac{1}{3}xy^3$

25. $(-x^4y)(-xy^4)$.

26. $-18x^3y^2$ / $-\frac{1}{6}xy^2$

27. $\frac{5}{6}$ / $12x^4$

28. $-8xy$ / $+\frac{1}{2}xy$

29. $x^4 \cdot x^{12}$.

30. $2^4 \cdot 2^2$.

State the value of each of the following:

31. $(3xy)(-4x^2y)(+2xy^2)$.

32. $(2x^2)^2$.

33. $(4x^3)^2$.

34. $(5x^4)^2$.

35. $(-3x)^2$.

36. 5 times what $= -30x$.

37. ? times $7x = 35xy$.

38. ? times $3x^2 = -12x^3y$.

39. $5xy^2$ times ? $= 35x^3y^3$.

40. $3x^2$ times $5x^2$ times ? $= 60x^5$.

Optional

Multiply each of the following:

41.

$-7x^2y$	$3ab^4$	$10m^3$	$-6x^4y^5$
$4xy^2$	$2a^6$	$-2m^2n^4$	$-2xy^4$

42. $(-4a^3b^3)(-10ab^5)$; $(6a^2b)(-10)$; $(-4a)(6a^2bc)$.

43.

$3x^2$	$\frac{1}{2}x^2$	$.4x$	$3\frac{1}{2}x^4$	$\frac{1}{2}x^5$
$.4x^3$	$\frac{1}{3}x^3$	$.02x$	$\frac{1}{4}x^2$	$.5x^2$

44.

2^5	2^{n+1}	2^{n-1}	x^{n+1}	x^{n-4}
2^2	2^2	2^3	x^3	x^2

45.

$2(a+b)^2$	$(a-b)^3$	$(x+y)^2$	$4(x-y)$
$3(a+b)^4$	$(a-b)^5$	$5(x+y)^3$	$3(x-y)^3$

46. Product of Polynomial and Monomial. — The multiplication of a polynomial and monomial is performed by the following

Rule. — *Multiply each term of the polynomial in succession by the monomial.*

Illustrative example:

Multiply $3x^3 - 6x^2 + 8x - 3$ by $-4x^2$.

Solution:

$$\begin{array}{r} 3x^3 - 6x^2 + 8x - 3 \\ -4x^2 \\ \hline -12x^5 + 24x^4 - 32x^3 + 12x^2 \end{array}$$

The work may also be indicated with a parenthesis thus:

$$-4x^2(3x^3 - 6x^2 + 8x - 3) = -12x^5 + 24x^4 - 32x^3 + 12x^2.$$

As shown in section 2, the multiplication is here indicated by the multiplier, $-4x^2$, standing adjacent to the parenthesis with no sign between.

Similarly, $-(x^2 + 3x - 2)$ may be regarded as

$$-1(x^2 + 3x - 2),$$

which equals $-x^2 - 3x + 2$.

In like manner the multiplication of two polynomials may be indicated by inclosing them in adjacent parentheses.

Exercises

Multiply:

1. $4(x + 3y)$.
2. $3x(x + 2)$.
3. $5x(x - y)$.
4. $2x(y + z)$.
5. $-3x(y - 2z)$.
6. $x(x^3 + 2x^2 + 3)$.
7. $3x^2(x^4 - 2x^3)$.
8. $-5x^5(2x^2 - 3x)$.
9. $(5 + 2x - 3x^3)3x^4$.
10. $(3x^3 - 4x^2 - 2)(-2x^2)$.
11. $-3ab(6a^3 + 5a^2b - 4ab^2 + 6b^3)$.
12. $(x^3y^3 - 4x^2y^2 - 10xy - 5)(10xy^2)$.

13. $3x^4 + 6x^3 - 7x^2 + 8x - 2$ by $-3x^2$.

14. $-4y^3 + 11y^2 - 8y + 1$ by $6y$.

15. $B^2 - 4AC$ by $6AC$.

16. $A^2 + 2AB + B^2$ by $4A^2B^2$.

17. $\frac{1}{2}(6x^2 + 4x - 2)$.

18. $(-\frac{1}{4}R)(\frac{1}{2}R^2 - \frac{1}{3}RD + \frac{1}{6}D^2)$.

19.
$$\begin{array}{l} \frac{1}{2}x^2 + \frac{1}{3}xy + \frac{1}{2}y^2 \\ \frac{1}{2}xy \\ \hline \end{array}$$

20.
$$\begin{array}{l} 3.25a^3 + .03a^2 - 1.4a + 6 \\ .4a^2 \\ \hline \end{array}$$

Simplify the following by removing the parentheses and combining (adding) similar terms:

21. $4(2x + y) - 4x$.

22. $6(R - D) + 2(R + D)$.

23. $5(x^2 + 2xy + y^2) + 4(2x^2 - xy + y^2)$.

24. $2(4H - 3K) - 2(3H - 4K)$.

25. $\frac{1}{2}R(R^2 - \frac{1}{4}R) + \frac{1}{4}(8R^2 + \frac{1}{2}R)$.

Time-Practice Test. 3 Minutes

Use of Parentheses in Multiplication

Write results.

1. $5(3x + 2)$.

2. $3(4x - 5)$.

3. $6(x - 3y)$.

4. $-7(x + 3)$.

5. $-4(x + 9)$.

6. $-1(x + 3)$.

7. $-1(x - 3)$.

8. $-1(-x + 5)$.

9. $-3x(3 - 2x)$.

10. $-7x(-x - 20)$.

11. $5(-x + 4)$.

12. $4x(3x + 2)$.

13. $6x(-x - 7)$.

14. $7(a - 2b)$.

15. $3(x - 3y)$.
16. $5x(a - 4)$.
17. $15(-3 + x)$.
18. $9x(-x + 5)$.
19. $-4x(-4 + x)$.
20. $-1(x + y - z)$.
21. $-3(2x - 3y + 4z)$.
22. $-5(3x^2 - 4x + 6)$.
23. $7(x^2y - y)$.
24. $-8(x^3y - 2x^2 - 6)$.
25. $\frac{1}{2}(4x^2 + 6x + 8)$.

47. Multiplication of a Polynomial by a Polynomial is performed according to the following

Rule.—(1) *Arrange both the multiplicand and the multiplier in either ascending or descending powers of some common letter.*

(2) *Multiply the entire multiplicand by each of the terms of the multiplier in succession, beginning at the left.*

(3) *Place similar terms in vertical columns and add the partial products.*

Illustrative example I:

Multiply $2x + 4$ by $4x + 8$, and check by letting $x = 2$.

Solution:

$$\begin{array}{rl} 2x + 4 & = 4 + 4 = 8 \\ \underline{4x + 8} & = 8 + 8 = \underline{16} \\ 8x^2 + 16x & \qquad 128 \\ \quad + 16x + 32 & \\ \hline 8x^2 + 32x + 32 & = 32 + 64 + 32 = 128 \end{array}$$

Illustrative example II:

Multiply $x^2 - x^3 + 3 - 5x$ by $x - 1 + 4x^2$.

Solution:

$$\begin{array}{rl} -x^3 + x^2 - 5x + 3 & = -8 + 4 - 10 + 3 = -11 \\ 4x^2 + x - 1 & = \quad 16 + 2 - 1 = \underline{17} \\ \hline -4x^5 + 4x^4 - 20x^3 + 12x^2 & \qquad -187 \\ \quad - x^4 + x^3 - 5x^2 + 3x & \\ \quad + x^3 - x^2 + 5x - 3 & \\ \hline -4x^5 + 3x^4 - 18x^3 + 6x^2 + 8x - 3 & = -187 \end{array}$$

Explanation: Arrange terms in descending powers. Multiply by $4x^2$, then by x and then by -1. Now add.

Check of Multiplication. —

In the above example let $x = 2$.

Then the multiplicand becomes -11.

The multiplier becomes $+17$.

The product becomes $-128 + 48 - 144 + 24 + 16 - 3 = -187$.

Since the product of -11 by $17 = -187$ the multiplication is correct.

Exercises

Multiply and check as directed:

1. $x + 2$ by $x + 1$.
2. $x - 2$ by $x - 1$.
3. $x + 2$ by $x - 1$.
4. $x - 2$ by $x + 1$.
5. $x + 4$ by $x + 2$.
6. $x + 5$ by $x + 2$.
7. $x + 3$ by $x + 2$.
8. $x + 2$ by $x + 6$.
9. $x - 3$ by $x + 2$.
10. $x + 4$ by $x - 2$.
11. $x + 3$ by $x + 4$.
12. $x - 4$ by $x - 3$.
13. $x + 4$ by $x - 3$.
14. $x + 6$ by $x + 8$.
15. $(x + 8)(x - 4)$.
16. $(2x + 4)(x + 5)$.
17. $(4x - 3)(4x + 2)$.
18. $(7x - 4)(2x - 5)$.
19. $(3a - 5)(4a + 8)$.
20. $(2x - 4)(3x + 5)$.
21. $(2x - y)(x - 2y)$.
22. $(3a + 2b)(2a - 3b)$.
23. $(5h - 2k)(2h + 3k)$.
24. $(8x + 4y)(5x - 6y)$.
25. $(7x - 3y)(3x + 7y)$.
26. $(h + 5k)(h - 7k)$.
27. $(3x + y)(5x - 6y)$.
28. $4h - 5k$ by $2h + 3k$.
29. $8 - 5y$ by $8 + 4y$.
30. $3h - k$ by $2h + 5k$.
31. $5x - 4y$ by $4x - 5y$.
32. $x^2 - 9$ by $x^2 - 3$.
33. $x^2 + 2y^2$ by $2x^2 + 3y^2$.
34. $(x^2 + 7y^2)(x^2 - 4y^2)$.

35. $(2x^2 + 5y^2)(3x^2 - 2y^2)$.

36. $(7x^2 - y^2)(2x^2 + 3y^2)$.

37. $(7 - 3m^2)(3 + 5m^2)$.

38. $(5m^2 - n)(3m^2 + 2n)$.

39. $(6xy - 5)(3xy + 2)$.

40. $x^2 + 2x + 3$ by $x + 2$.

41. $x^2 + 2x + 1$ by $x + 1$.

42. $x^2 + 5x + 6$ by $x + 3$.

43. $2x^2 + 3x + 4$ by $2x + 3$.

44. $4x^2 + x + 5$ by $x + 2$.

45. $(6x^2 + 3x^3 - 4x + 2)(x^2 + 1)$.

46. $(6mn + 4m^2 - 2n^2)(3m + 4n)$.

47. $(8x^2 - 7x^3 + 1 - 4x)(1 - 4x + 2x^2)$.

48. $(3x^3 - 1 + 2x^5)(3x^2 - 4)$.

49. $(x^2 + xy + y^2)(x - y)$.

50. $3x^2 - 4x + 2$ by itself.

Optional

51. $(4a^2 + 6a - 7)(a^2 - a + 5)$.

52. $(-6a^2 + a + 2a^3 - 1)(2a + a^2 - 5)$.

Find the value of:

53. $(2a^2 - a + 4)^2$.

54. $3x^3 + 6x^2 - (2x - 1)^3$.

55. Multiply $x^2 + 2x + 3$ by $x^2 + x - 3$.

56. Multiply $\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{3}$ by $x - 2$.

57. $(2C - 1)(C + 2)(3C - 2)$.

58. $(\frac{1}{2}x + \frac{1}{3})(\frac{1}{2}x - \frac{1}{3})$.

59. $(.5x - 4y)(.2x - .3y)$.

60. What is the shortest way to express the product of $(x + 2)(x + 2)(x + 2)$?

61. Show by the accompanying diagram that

$$(x + y)(z + w) = xz + yz + xw + yw.$$

	x	y
z	xz	yz
w	xw	yw

62. Show by a diagram similar to that in example 61, that

$$(x + 4)(x + 5) = x^2 + 9x + 20.$$

48. Equations Involving Multiplication. —

Illustrative example I:

Solve the equation $4(x + 3) = 20$.

Solution: One way to solve this type of an equation is to multiply $x + 3$ by 4. Then we have $4x + 12 = 20$.

Complete the solution and check in the original equation.

Illustrative example II:

Solve $10 - 2(x - 2) - 4x = 44$.

Solution: $10 - 2x + 4 - 4x = 44$.

$14 - 6x = 44$. Adding like terms

Complete the solution.

Exercises

Solve the following equations and check as directed:

1. $3(x + 3) = 12$.
2. $2(x + 4) = 16$.
3. $4(x - 5) = 20$.
4. $6(x - 4) = -30$.
5. $-2(x + 10) = 40$.
6. $-1(x + 12) = 8$.
7. $-(14 - x) = 20$.
8. $-(7 - 3x) = 14$.
9. $-3(x - 2) = 18$.
10. $-4(5 - x) = 20$.
11. $-6(3 - x) = 30$.
12. $+3(4x - 1) = 21$.
13. $20 = -5(3x - 1)$.
14. $40 = -2(x - 4)$.
15. $60 = -3(x - 1)$.

Review Exercises in Algebraic Expression

Many of these questions may be answered orally.

1. If x is one side of a square, what is its perimeter? Its area?

2. If $x + 3$ is the length of a rectangle and x is its width, what is its perimeter? Its area?

3. Write as an equation the statement that the area of the rectangle in example 2 is 30 more than that of the square in example 1.

4. The sum of two numbers is 10 and one of them is 6. What is the other? Write answer using 10, 6, and a minus sign.

5. The sum of two numbers is 10 and one of them is n. What is the other?

6. The product of two numbers is $10\,x^3$ and one of them is x. What is the other?

7. The difference of two numbers is 10 and the smaller is x. What is the larger? If the larger number is x, what is the smaller? Write with correct sign.

8. If a man can walk 3 miles an hour, how far does he walk in 15 hours? In n hours? In $x + 1$ hours?

9. If an automobile can run 250 miles in 5 hours, how fast is that in miles per hour?

10. If an automobile can run x miles in 4 hours, how fast is that in miles per hour?

11. A train went a distance of $x + 1$ miles at the rate of a miles per hour. How many hours were required?

12. If your age is x years and your cousin is twice as old, how old is he? How old will you be in 10 years? What will be his age in 10 years?

13. If x denotes a certain number, what will denote a number 10 less? 15 more? 12 times as large? $\frac{1}{3}$ as large?

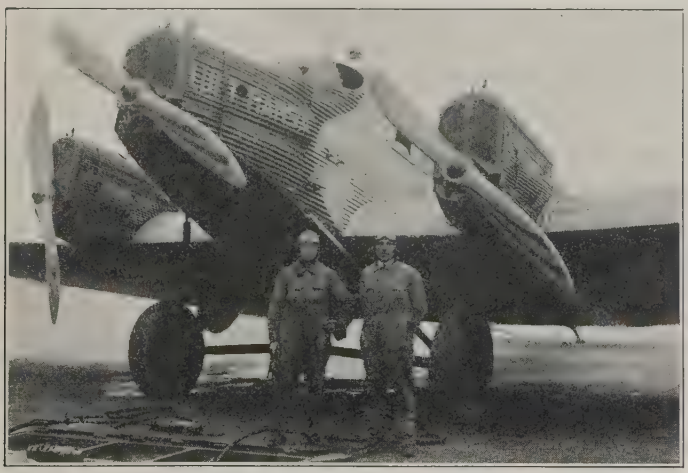
14. If x represents the number of papers sold by one newsboy on a certain day and another newsboy sells 15 more than twice as many as the first, express the number sold by the second boy.

15. What is the excess of 60 over x? Of x over 60?

16. By how much does y exceed 15?

17. Express the interest on d dollars at p per cent for y years.

18. If I have d dollars and you have \$5 more than three times as much, how much have you? How much have we both?



A German Airplane

19. If an airplane averages 90 miles per hour, how far will it travel in 10 hours? If it averages m miles per hour? If it averages $m + 1$ miles per hour?

20. How far can I walk in h hours at the rate of m miles per hour?

21. Draw a square. Mark one side of it 6. Write the area inside of the square. What is the perimeter of the square?

22. Draw a square. Mark one side of it s. Write its area inside the square. What is its perimeter?

23. A boy has d dollars and c cents. How many cents has he?

24. What is the cost in dollars of 25 articles at d dimes each?

25. How much larger is $h + k$ than $2\,k$?

26. Write in algebraic language the fact that the product of x and y subtracted from the product of $-\,2\,x$ and y equals $-\,3\,xy$.

27. Write with algebraic symbols: six times the product of y and the fifth power of x equals one hundred ninety-two.

28. Write with symbols: the fourth power of x multiplied by the fifth power of y and the result added to the sixth power of z.

29. If I have a line 10 feet long and mark one part of it y, what will express the length of the other part?

30. If there are 4300 pupils in our school and there are b boys, what will express the number of girls?

49. The Solution of Problems. — In solving applied problems you must use a certain amount of independent reasoning. No definite rules can be given. You should read again the introduction to exercises on page 44.

Usually it is best to let x or some other letter represent the unknown quantity. Read the problem until you thoroughly understand its meaning and the relations of the numbers involved in it. Then state this relation in equation form. Also see page 168.

Number Problems

1. The sum of two numbers is 20. Five times the first number plus eight times the second number equals 127. What are the numbers?

Hint: Let x = one number.

$20 - x$ = other number.

$5x + 8(20 - x) = 127$. Complete.

2. One number exceeds another by 6, and 5 times the first number exceeds 6 times the second number by 6. What are the numbers?

3. Divide 40 into two parts such that one part increased by 10 times the other part shall equal 76.

4. Divide 45 into two parts such that if the greater be added to 20, and the less to 35, the resulting numbers shall be equal.

5. I bought 6 pounds of coffee and 7 pounds of tea for $6. The tea cost 30 cents a pound more than the coffee. What was the cost of each per pound?

Coin Problems

6. The value of 60 coins is $5. The coins are dimes and nickels. How many are there of each?

Hint: Let d = number of dimes.

Then, $60 - d$ = number of nickels.

Then study the following tabulation

Kind	Numbers	Value in Cents
Dimes	d	$10d$
Nickels	$60 - d$	$5(60 - d)$

Therefore, $10d + 5(60 - d) = 500$. Complete.

7. A sum of $7 is made up of 46 coins, consisting of quarters and dimes. How many are there of each?

8. The value of 72 coins is $5.90. The coins are dimes and nickels. How many are there of each?

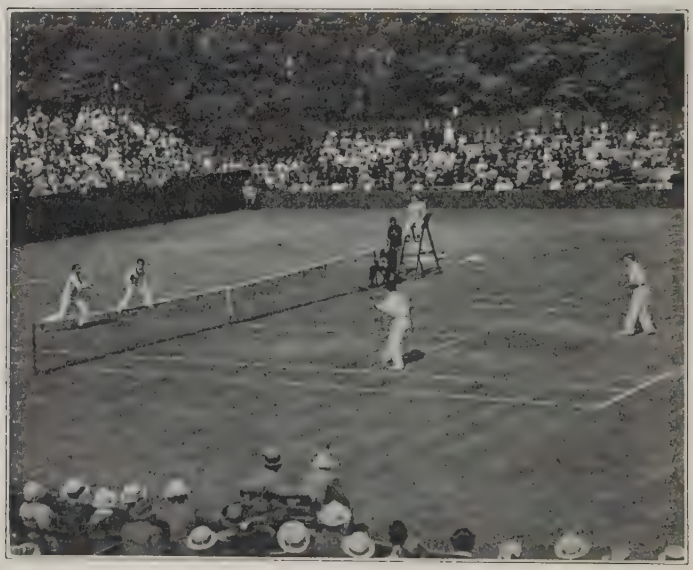
9. \$6.25 is made up of 3 times as many quarters as half-dollars. How many are there of each?

10. \$10.50 is made up of an equal number of dimes and quarters. How many are there of each?

Geometry Problems

11. If a rectangle is four times as long as it is wide and its perimeter is 60 feet, find the length and width. Draw figure.

12. Divide a pole 32 feet long into two parts so that 3 times the longer part shall equal 5 times the shorter part.



A Championship Match

13. A tennis court for four players is 6 feet longer than twice its width. The perimeter of the court is 228 feet. Find the dimensions of the court.

14. Find where to cut a board which is 46 inches long, into two parts such that their difference is 30 inches.

15. A piece of iron 52 inches long is to be cut into two pieces so that one part exceeds the other by 4 inches. Find the length of each part.

16. One side of a triangle is two times as long as the shortest side. The other side is 8 feet longer than the shortest side. If the perimeter of the triangle is 38 feet, find the length of each side.

17. A line 90 inches long is divided into two parts. Three times the smaller part is 60 inches less than twice the larger part. How many inches in each part?

18. A line 50 inches long is divided into two parts such that four times the smaller part is 25 less than the larger part. Find each part.

Age Problems

19. Mary's age is four times Esther's, and the sum of their ages exceeds their difference by 12. Find their ages.

20. John is twice as old as James. 12 years ago he was three times as old. How old is each?

Let x = James's age now.

Then $2x$ = John's age now.

	Age Now	Age 12 Years Ago
James . .	x	$x - 12$
John . .	$2x$	$2x - 12$

Therefore $2x - 12 = 3(x - 12)$.

21. Fourteen years ago John was twice as old as his sister and now the sum of their ages is 40 years. What is the age of each?

22. The age of the older of two boys is twice that of the younger; three years ago it was three times that of the younger. Find the age of each.

Multiple Choice or Selection Exercises

From the expressions on the right of the equality sign select the correct answer to the expression to the left of the sign.

1. $-x^3 \cdot x^5 = x^8, -x^{15}, -x^8, x^{15}.$
2. $-5(x+3) = -5x+3, x-15, -5x-3, -5x-15.$
3. $-6(x-7) = -6x-7, x-42, -6x+42, -6x-42.$
4. $-3(4x-2) = -12x-2, 4x+6, -12x+6, -12x-6.$
5. $-3x^4 \cdot 5x^2 = 15x^6, -15x^8, -15x^6.$
6. $3(3x-4) = 9x-12, 9x-4, 9x+4, 3x-12.$
7. If $x = 3$, then $4(2x+3) = 9, 27, 36, 24.$
8. $-x^3(4x-x^3) = -4x^4-x^3, -4x^4+x^6, -4x^4+x^3, 4x^4+x^6.$
9. If $x = 8$, then $2x^2 - x = 64, 56, 120.$
10. $-5x^6 \cdot x^3 = -5x, -5x^9, 5x^{18}.$
11. $xy - z = 1, 11, -13$ when $x = 2$, $y = 3$, and $z = -5.$
12. If $x = -7$, then $5(x-8) = -35, 75, -75.$

Review

1. How long will it take a man to drive 120 miles at an *average* of 24 miles an hour while driving, if he makes three stops averaging 12 minutes each?

State which of the next five examples are true and which are false.

2. The expression n^2 means the same as $2n$.
3. If $x - 4 = 7$, then $x = 11$.
4. If $a = 6$ and $b = 7$, then $\frac{a}{b} \cdot \frac{b}{a} = 0$.
5. The number which is two more than a is b.
6. The number which is 5 less than n is $n - 5$.

Copy and complete each of the following statements:

7. The cost of a ham depends upon the . . . and the . . . per pound.

8. The number of books (same kind) which can be bought with a \$5 bill depends upon the . . . of . . . book.

9. The value of the expression $6n + 5$ depends upon the value of . . .

10. My total coal bill for a season depends upon the number of . . . and the . . . per ton.

11. The interest that I receive upon \$1000 for 1 year depends upon . . .

12. Add and check: $5y^2 - 10yz + 3z^2$, $7y^2 - 8z^2$, $9y^2 + 9yz + 5z^2$.

13. Solve the following equations:

(a) $9x - 5(x - 4) = 28$. (c) $x(x+3) = x(x+4) - 5$.

(b) $7x + 4(x - 2) = 25$. (d) $\frac{3}{4}x + 7 = 13$.

14. Is the square of any number always greater than the number? Prove.

15. Add:
$$\begin{array}{r} 2x + 3y \\ 2x + 3y \\ \underline{2x + 3y} \end{array}$$

16. Add:
$$\begin{array}{r} 4x^4 - 3x^2 - 6x - 2 \\ -6x^4 + 5x^2 + 8x - 6 \\ \underline{5x^4 - 7x^2 + 4x - 9} \end{array}$$

17. From the sum of $3x^3 - 4x^2 - 5x + 8$, $2x^2 + 4x - 6$, and $9x^2 + 6x - 12$ subtract the quantity $4x^3 - 7x^2 + 10x + 10$. Prove your work.

18. $4(x^2 - y^2) + 9(x^2 - y^2) - 5(x^2 - y^2) + 12(x^2 - y^2) = ?$

19. From the sum of $3x + 4y$ and $4x - 3y$, subtract four times $(2x - 4y)$, and check with $x = 1$ and $y = 2$.

20. What must be subtracted from $3x + 4y^2 - 5z$ to produce a remainder of $2x + 3y^2 + 5z$?

21. By what does the sum of $6m^2 + 4m - 3$ and $10m^2 - m - 6$ exceed $2m^2 - 3m - 9$?

22. From $3x^3 + 5x^2 + 12x - 14$
subtract $6x^3 - 7x^2 - 15x + 24$

23. Multiply $x^2 + 3x + 4$ by $2x - 3$.

Simplify:

24. $(x + 2)(x + 3) + (2x - 1)(x - 4)$.

25. $(H + K)(H + K) - (H - K)(H - K)$.

26. $x^2 - 3x + 4 - (2 - x)(3x + 2)$.

27. Find the value of

$$(x + y)(x - y) - \frac{xy + xz + xw}{k}$$

if $x = 6$, $k = 4$, $y = 3$, $w = 2$, and $z = 1$.

28. If $x = 4$, write the right-hand member of the equations:
(a) $4x - 5 =$ (b) $6x + 6 - 2x =$ (c) $4 - 3x + 5x =$

29. If t stands for one tenth and h for one hundredth, express $1 - 4t + 5h$ in per cent.

Express 56% in terms of t and h.

30. Write as an algebraic formula: The product equals the multiplicand times the multiplier.

31. Solve and check:

(a) $5(x - 2) = 20$.
(b) $6(x - 1) = 24$.
(c) $3(x - 5) = 30$.
(d) $4(x - 3) + 6 = 18$.
(e) $3(x - 7) + 18 = 3$.
(f) $4(x - 2) - 7 = 37$.

32. The average weight of boys at different ages, beginning at 6 years and continuing to 15 years by one authority, is given in the table below:

Age	6	7	8	9	10	11	12	13	14	15
Weight (in lbs.)	50	53	57	62	67	72	78	85	93	105

(a) Make a graph of this table.

(b) A boy, normal in every respect, weighs 87 pounds. What is his approximate age? Show how this approximation is made from the graph.

(c) The weight increases as . . .

50. Division of Monomials. — The law of signs in division has been stated in section 29.

The law governing the use of exponents can be inferred from the following:

$$\frac{a^7}{a^5} = \frac{\not{a} \cdot \not{a} \cdot \not{a} \cdot \not{a} \cdot \not{a} \cdot a \cdot a}{\not{a} \cdot \not{a} \cdot \not{a} \cdot \not{a} \cdot \not{a}} = a^2.$$

Dividing both numerator and denominator by like quantities, we obtain as a result a^2.

We may also infer this result from the rule for multiplication, for since

$$a^2 \cdot a^5 = a^7, \text{ then } \frac{a^7}{a^5} = a^2, \text{ and } \frac{a^7}{a^2} = a^5.$$

How is the exponent 2 in the quotient obtained from the exponents 7 and 5? Can you state the rule in words?

In general, then, we write:

$$\frac{a^m}{a^n} = a^{m-n}.$$

Rule. — *To divide one monomial by another,* (1) *divide the coefficient of the dividend or numerator by the coefficient of the divisor or denominator, observing carefully the law of signs, thus obtaining the coefficient of the quotient.*

(2) *Subtract the exponents of letters in the divisor or denominator from those of like letters in the dividend or numerator to obtain the exponent of those letters in the quotient.*

(3) *Omit any letter which has the same exponent in the dividend and divisor.*

(4) *If a letter occurs in the dividend which is not found in the divisor, it is a part of the quotient.*

(5) *If a letter occurs in the divisor and not in the dividend it is in the denominator of the result.*

Illustrative examples:

$$\frac{32\,a^5}{-\,8\,a^2} = -\,4\,a^3; \quad \frac{-\,20\,x^5y^3}{-\,4\,x^2y^2} = 5\,x^3y; \quad \frac{-\,40\,x^3y^2}{5\,x^3y} = -\,8\,y;$$

$$\frac{24\,n^2m^2}{4\,n} = 6\,nm^2; \quad \frac{-\,3\,x^{n+2}}{x^n} = -\,3\,x^2; \quad \frac{10\,x^2y^3}{5\,xy^2z} = \frac{2\,xy}{z}.$$

Oral Exercises

1. How many packages of \$6 each would \$18 make?

2. Some boys divided a dozen oranges so that each received four. How many boys were there?

State the quotient of each of the following:

3. 3)24 quarts.
4. 3)12 x.
5. $32\,xy \div 4$.
6. $30 \div (-\,3)$.
7. $(-\,30) \div (-\,3)$.
8. $-\,45 \div (-\,15)$.
9. $x^4 \div x^2$.
10. $x^5 \div x^3$.
11. $x^8 \div x^5$.
12. $x^8 \div x^3$.
13. $x^7 \div x^4$.
14. $12\,x^5 \div 3\,x^3$.
15. $-\,30\,x^4 \div (-\,10\,x)$.
16. $27\,y^5 \div (-\,9\,y^3)$.
17. $24\,x^9y^6 \div 4\,x^9y^4$.
18. $16\,x^3y^5 \div (-\,8\,xy^4)$.
19. $-\,28\,xy^3 \div 7\,xy^2$.
20. $x^{12} \div x^9$.
21. $2^{12} \div 2^9$.
22. $3^5 \div 3^2$.

Exercises

Divide:

1. x^8 by x^4.
2. $-\,5\,x^4$ by x.
3. $8\,a^2b$ by $2\,a$.
4. $-\,16\,a^4b^3$ by $8\,a^3b$.

5. $-10\,x^5y$ by $-2\,x^3y$.
6. $18\,a^3b^2c$ by $-2\,abc$.
7. $9\,xyz^3$ by $9\,xz^2$.
8. $-12\,x^2y^2z^2$ by $6\,xyz$.
9. $-24\,a^3bc^2$ by $-6\,abc$.
10. $-12\,a^3bk^2$ by $-4\,a^3k$.
11. $-8\,x^3y)\underline{16\,x^4y^4}$.
12. $5\,mn)\underline{10\,m^2n}$.
13. $-15\,a^3b^3cd$ by $5\,a^3b^2d$.
14. $12\,m^4n^4$ by $-2\,m^3n^3$.
15. $100\,a^3x^3y^2$ by $-10\,ax^3y$.
16. $-50\,a^2x^4z^3$ by $5\,ax^3z^3$.
17. $64\,a^2x^2m^3n$ by $8\,axmn$.
18. $121\,x^4y^3z$ by $-11\,xy^3z$.
19. $-5\,x^3$ by -1.
20. $-10\,x^4$ by -2.
21. $90\,xy$ by $9\,xz$.
22. $16\,bx^2$ by $4\,ax^2$.
23. $-16\,xy$ by $\frac{1}{2}\,y$.
24. $-16\,xy$ by $\frac{1}{2}\,x$.
25. $-7\,x$ by $-14\,x$.

Optional

26. $-\frac{1}{8}\,\pi r^2$ by $\frac{1}{4}\pi r^2$.
27. $3\frac{1}{4}\,y^4$ by $\frac{1}{2}\,y^3$.
28. $.9\,HK^2$ by $.3\,K^2$.
29. $.02\,HK$ by $.4\,HK$.
30. $-.42$ by $-.14$.
31. $.6\,H^2K^2$ by $.3\,H^2$.
32. $6(a+b)$ by $2(a+b)$.
33. x^a by x^b. *Solution:* $x^a \div x^b = x^{a-b}$.
34. e^x by e^y.
35. a^m by a^n.
36. $12\,x^{3a}$ by $3\,x^a$.
37. $24\,x^{6a}$ by $6\,x^{4a}$.
38. $18\,e^{a+3}$ by $6\,e^3$.
39. $-36\,x^{3m+4}$ by $9\,x^{2m+1}$.
40. $-48\,x^{5m}y^{3n}$ by $-16\,x^{3m}y^n$.

51. Division of Polynomials by Monomials. — **Illustrative example:**

$$\frac{24\,x^4 - 16\,x^3 + 8\,x^2}{8\,x} = 3\,x^3 - 2\,x^2 + x.$$

Check: $8\,x(3\,x^3 - 2\,x^2 + x) = 24\,x^4 - 16\,x^3 + 8\,x^2$.

Rule. — *To divide a polynomial by a monomial divide the terms of the polynomial successively by the monomial, beginning at the left.*

Exercises

Divide and check:

1. $\dfrac{x^4 - x^3}{x}$.
2. $xy + xz$ by x.
3. $8\,ab + 10\,a^2b^2$ by $2\,ab$.
4. $8\,x^3 - 4\,x^2y - 16\,x^2$ by $4\,x^2$.
5. $B^3 - 3\,B^2$ by B^2.
6. $6\,x^3y^2 + 16\,xy^3 + 24\,x^2y^2$ by $2\,xy$.
7. $-a - b + c$ by -1.
8. $6 - 3\,x - 4\,x^2$ by -1.
9. $-8\,m^2n^4 + 24\,m^3n^2 - 32$ by -8.
10. $(16\,x^4 + 64\,x^3y^2 - 48\,x^3) \div (-8\,x^3)$.
11. $(27\,a^3b^3 + 81\,ab^4 - 9\,a^2b^5) \div (-9\,ab^3)$.
12. $(-21\,y^4 - 35\,y^3 - 14\,y^2) \div (-7\,y^2)$.
13. $(8\,x^4 + 12\,x^3 - 16\,x^5) \div (4\,x)$.
14. $(-12\,m^2n - 16\,mn^2 + 24\,mn^3) \div (4\,mn)$.
15. $(H^2K^4 - 9\,HK^3) \div (-HK^3)$.
16. $(25\,x^3y^2 - 40\,xy^3 - 15\,x^3y^3) \div (5\,xy)$.
17. $(-88\,a^4x^3y + 77\,a^3xy^2 - 44\,a^5x^2y) \div (-11\,a^2xy)$.
18. $6\,ab^2)36\,a^3b^4 - 60\,a^4b^3 - 12\,a^3b^2 + 30\,ab^2$.

19. If we divide an empty tin box into 6 compartments by the use of partitions, will each compartment be empty?

20. $0 \div 6 = ?$ $\quad 0 \div 8 = ?$ $\quad 0 \div 10 = ?$
$0 \div 3\,x = ?$ $\quad 0 \div 5\,k = ?$ $\quad 0 \div 100 = ?$

Optional

21. $\dfrac{(h + k)y + (h + k)y^2}{h + k}$.

22. $\dfrac{6(y + 1) + h(y + 1) + k(y + 1)}{y + 1}$.

23. $\frac{1}{2}\,x)\frac{1}{2}\,x^2 + \frac{1}{4}\,x.$

24. $(\frac{1}{3}\,x^2 + \frac{1}{2}\,x^3 + \frac{1}{4}\,x^5) \div \frac{1}{2}\,x^2.$

25. $.3\,x^3).3\,x^3 + 1.8\,x^4 + 1.2\,x^8.$

26. $(.5\,x + x^2 - 1.5\,x^3) \div (.5\,x).$

27. $4\,x)8\,x^4 + 12\,x^3 - 16\,x^5.$

28. $4\,x^n)8\,x^{4n} + 12\,x^{3n} - 16\,x^{5n}.$

29. $x^n)x^{n+2} + 2\,x^{n+1} + 4\,x^{n-1} - 6\,x^n.$

30. $-5\,ab)1.5\,a^2b + 2\,ab^2 - 2.5\,a^3b^3.$

31. What power of y is the quotient of $y^8 \div y$? Of $\frac{y^5}{y}$? Of $y^{12} \div y^7$?

True-False Exercises

Copy and place a *t* after those that are right and an *f* after each one that is incorrect:

1. $\frac{x^3}{x} = x^2.$ true

2. $\frac{x^4}{x^2} = x^8.$ false

3. $\frac{-x^8}{x^2} = -x^4.$ false

4. $\frac{x^3}{x^3} = 0.$ false

5. $\frac{-x^4}{x^3} = -x.$ true

6. $\frac{-x^7}{-x^7} = -1.$ false

7. $\frac{x^8}{x^8} = 1.$ true

8. $\frac{16\,x - 8}{8} = 2\,x - 1.$ true

9. $x^4 \cdot x^2 = x^8.$ false

10. $7\,x^3 \cdot 9\,x^4 = 63\,x^{12}.$ false

11. $\frac{x^2 + x}{x} = x + 1.$ true

12. $\frac{35\,x^7}{7\,x^5} = -5\,x^2.$ false

13. $\frac{42\,x^7 - 28\,x^5}{7\,x^3} = 6\,x^4 - 4\,x^2.$ true

14. $-7\,x^5 \cdot 4\,x^2 = -28\,x^{10}.$ false

15. If $x = 2$, then $x^3 - 4\,x^2 = -8.$ true

52. Division of a Polynomial by a Polynomial. —

Rule. — (1) *Arrange both the dividend and divisor according to either the ascending or descending powers of a common letter.*

(2) *Obtain the first term of the quotient by dividing the first term of the dividend by the first term of the divisor.*

(3) *Multiply the entire divisor by the first term of the quotient and subtract the result from the dividend, bringing down all the terms of the dividend.*

(4) *Obtain the second term of the quotient by dividing the first term of the remainder by the first term of the divisor.*

(5) *Multiply the entire divisor by the second term of the quotient, subtract, and so continue until there is either no remainder or until the remainder can no longer be divided by the divisor.*

Illustrative example I:

Divide $4x^2 + 8x + 3$ by $2x + 3$.

$$\begin{array}{rl|l} 483 & 23 & 4x^2 + 8x + 3 \;\big|\; 2x + 3 \\ \underline{46} & \underline{21} & \underline{4x^2 + 6x} \qquad\quad 2x + 1 \\ 23 & & 2x + 3 \\ \underline{23} & & \underline{2x + 3} \end{array}$$

Check: $(2x + 3)(x + 1) = 4x^2 + 8x + 3$.

Illustrative example II:

Divide $6 + 12x^2 - 19x + 4x^3$ by $2x - 1$.

Solution:

	Dividend		
(a)	$4x^3 + 12x^2 - 19x + 6$	$2x - 1$	Divisor
(b)	$\underline{4x^3 - 2x^2}$	$2x^2 + 7x - 6$	Quotient
(c)	$14x^2 - 19x + 6$		
(d)	$\underline{14x^2 - 7x}$		
(e)	$-12x + 6$		
	$\underline{-12x + 6}$		

Explanation:

(a) Arranging the terms, by Rule (1).

(b) $4x^3 \div 2x = 2x^2$, by Rule (2); $2x^2(2x - 1) = 4x^3 - 2x^2$, by first part of Rule (3).

(c) Subtracting and bringing down terms, by last part of Rule (3).

(d) $14x^2 \div 2x = 7x$, by Rule (4); $7x(2x - 1) = 14x^2 - 7x$, by first part of Rule (5).

(e) Subtracting and continuing, by last part of Rule (5).

Illustrative example III:

Divide $2x^3 - 7x^2y + 10xy^2 - 10y^3$ by $x - 2y$.

Solution:

$$
\begin{array}{l|l}
2x^3 - 7x^2y + 10xy^2 - 10y^3 & x - 2y \quad \text{Divisor} \\
\underline{2x^3 - 4x^2y} & \underline{2x^2 - 3xy + 4y^2} \quad \text{Quotient} \\
\quad - 3x^2y + 10xy^2 & \\
\quad \underline{- 3x^2y + 6xy^2} & \\
\qquad + 4xy^2 - 10y^3 & \\
\qquad \underline{+ 4xy^2 - 8y^3} & \\
\qquad - 2y^3 \quad \text{Remainder} &
\end{array}
$$

This result may be written $2x^2 - 3xy + 4y^2 + \dfrac{-2y^3}{x - 2y}$ or, more simply, $2x^2 - 3xy + 4y^2 - \dfrac{2y^3}{x - 2y}$. The fraction must be connected to the polynomial by the proper sign. In arithmetic $6\frac{2}{3}$ means $6 + \frac{2}{3}$. Why must we use the + sign in algebra, although it is not necessary in arithmetic?

In subtracting, care must be exercised to keep the terms of each remainder arranged in the same order as those of the dividend and divisor.

Checks of Division. We can check our results in division as we did in other processes, by substituting numbers for the

letters. We can also check it by multiplication. If there is no remainder, the product of the divisor and quotient equals the dividend. If there is a remainder, this remainder added to the product of the divisor and quotient will give the dividend.

Exercises

Arrange in proper order. Divide. Check the first 5 by multiplication.

1. $x^2 + 10\,x + 21$ by $x + 3$.
2. $x^2 - 11\,x + 24$ by $x - 8$.
3. $x^2 - x - 12$ by $x + 3$.
4. $x^2 - x - 20$ by $x - 5$.
5. $x^2 + 2\,x - 35$ by $x + 7$.
6. $R^2 - 3\,R - 40$ by $R - 8$.
7. $x^2 + 13\,x + 36$ by $x + 9$.
8. $x^2 + 13\,x - 48$ by $x - 3$.
9. $x^2 - 24\,x + 44$ by $x - 2$.
10. $x^2 - x - 110$ by $x + 10$.
11. $2\,a^2 + a - 6$ by $a + 2$.
12. $4\,a^2 - a - 3$ by $a - 1$.
13. $2\,x^2 - 3\,x - 9$ by $x - 3$.
14. $-n^2 + 4\,n - 3$ by $n - 3$.
15. $-x^2 - 10\,x - 24$ by $x + 4$.
16. $x^2 - 4\,x - 12$ by $-x - 2$.
17. $-n^2 + 9\,n - 18$ by $-n + 3$.
18. $2\,x^2 - x - 15$ by $2\,x + 5$.
19. $2\,n^2 - 2\,n - 12$ by $2\,n - 6$.
20. $11\,x + 3\,x^2 - 20$ by $3\,x - 4$.
21. $-5\,y + 6\,y^2 - 6$ by $3\,y + 2$.
22. $19\,y + 10\,y^2 - 15$ by $5\,y - 3$.
23. $a^2 - 9$ by $a + 3$.
24. $a^2 - 9$ by $a - 3$.
25. $x^2 - 25$ by $x - 5$.
26. $x^2 - 25$ by $x + 5$.
27. $2\,a^2 - 8$ by $a + 2$.
28. $16\,x^2 - 1$ by $4\,x - 1$.
29. $8\,a^2 - 18$ by $2\,a + 3$.
30. $3 + 12\,a^2 - 20\,a$ by $2\,a - 3$.
31. $12\,a^2 + 10\,a + 2$ by $1 + 2\,a$.
32. $8\,a^2 - 22\,a + 15$ by $4\,a - 5$.
33. $21\,a^2 + 10 - 41\,a$ by $3\,a - 5$.
34. $21\,y^2 + 46\,y + 24$ by $6 + 7\,y$.
35. $6 - 16\,y + 8\,y^2$ by $-2\,y + 3$.
36. $-20\,m^2 + 64\,m + 21$ by $-2\,m + 7$.
37. $-6\,x^2 - 31\,x - 40$ by $-5 - 2\,x$.

38. $-22x - 8x^2 + 6$ by $2x + 6$.

39. $a^2 + 3ab + 2b^2$ by $a + b$.

40. $a^2 - 5ab - 14b^2$ by $a + 2b$.

41. $2x^2 - 7xy + 3y^2$ by $x - 3y$.

42. $12x^2 + 5xy - 2y^2$ by $2y + 3x$.

43. $2m^2 + mn - 3n^2$ by $2m + 3n$.

44. $-20b^2 + 7ab + 6a^2$ by $3a - 4b$.

45. $-18a^2 + 24ab + 10b^2$ by $-3a + 5b$.

46. $3a^2 - 11ab - 42b^2$ by $a - 6b$.

47. $x^3 + x^2 - 14x + 6$ by $x - 3$.

48. $x^3 + x^2 - 10x + 8$ by $4 + x$.

49. $x^3 - 7x^2 + 13x - 3$ by $x - 3$.

50. $3x^2 + 3x^3 + 12 - 14x$ by $x + 3$.

Optional

51. $x^3 - 1$ by $x - 1$.

52. $x^3 + 8$ by $x + 2$.

53. $x^3 - 27$ by $x - 3$.

54. $1 - x^4$ by $1 + x$.

55. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x - y$.

56. $-x^3y - 11xy^3 - 2x^2y^2 + 6x^4 - 6y^4$ by $2x - 3y$.

57. $x^4 + x^2 + 1$ by $x^2 - x + 1$.

58. $-53a^2 + 8 + 53a + 12a^3$ by $-1 - 7a + 4a^2$.

59. $a^{2x} + 2a^xb^x + b^{2x}$ by $a^x + b^x$.

60. $\dfrac{x^{3a} - y^{3b}}{x^a - y^b}$.

61. $\dfrac{27x^3 - 1}{3x - 1}$.

62. $\dfrac{8x^3 + 27y^3}{2x + 3y}$.

63. $\dfrac{x^2 + 4x + 3}{x - 4}$.

64. $\dfrac{.15x^2 - .02xy - .24y^2}{.5x + .6y}$.

65. $\dfrac{e^{3x} + 64}{e^x + 4}$

Matching Exercise

The answers in the right column are correct for the examples in the left column. Match them so that the proper answer is opposite the example to which it belongs.

1. $\frac{x^5}{x}$.	a. $5x^6$.
2. $x^3 \cdot x^4$.	b. $-10x^{10}$.
3. $5x^3 \cdot x^3$.	c. $-x^3$.
4. $x^7 \cdot x^3$.	d. $15x^8$.
5. $\frac{x^4}{x^4}$.	e. $-x^6$.
6. $(-5x^6)(-3x^2)$.	f. $-7x^2$.
7. $\frac{-x^9}{x^3}$.	g. x^7.
8. $\frac{-x^4}{x}$.	h. x^{10}.
9. $5x^7 \cdot -2x^3$.	i. 1.
10. $\frac{35x^7}{-5x^5}$.	j. x^4.

Multiple Choice Test

Select the expression from the parentheses which correctly supplies the missing words.

1. a^3 means ($a + a + a$, $a \cdot a \cdot a$, the cube root of a).

2. In the expression $3x$, 3 is a (coefficient, exponent, term).

3. In the expression $(a + b)+(x - y)$, $a + b$ is a (trinomial, term).

4. $a + b +(c - d)$ is a (monomial, binomial, trinomial).

5. $6\,a^2bcx(m+n)$ is a (monomial, binomial, trinomial).

6. $1-(-1)^3 = (1, 2, 0, 4)$.

7. The value of x in the equation $4x-(x-3)=2-x$ is $(1, -1, 4, -\frac{1}{4}, -2, \frac{1}{2})$.

8. The equation $\frac{x}{4} = 6$ is solved by (addition, subtraction, multiplication, division).

9. If $x = 3$, and $y = 4$, then $\frac{x+y-1}{2} = \left(\frac{x}{2}+\frac{y}{2}-\frac{1}{2},\ x+y-\frac{1}{2},\ \frac{x+y}{2}-1,\ x+\frac{y-1}{2}\right)$.

Review

1. Multiply $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ by $x-y$.

2. Add:

$$\begin{array}{r} 5(x^2+y^2+z^2) \\ -\,6(x^2+y^2+z^2) \\ \underline{-\,14(x^2+y^2+z^2)} \end{array}$$

3. If the length of a rectangle is represented by $x+4$ and the width is represented by $x+1$, and the side of a square is represented by x, find the area of each. Find the total area of both.

4. Evaluate your results in example 3 above when $x = 6$.

5. Subtract the quantity in the first column from the quantity in the second column.

1st Column	2d Column	Result
$3x+5y$	$2x-7y$	
$5x^2-6x$	$7x^2+6x$	
$8xy-7z$	$6xy+6z$	
$3x+5y-4z$	$2x-4y-2z$	

(a) Add. (b) Multiply.

6. From $12\,a^3 - 3\,a^2b - 5\,ab^2 - b^3$ take $3\,a^2b - 9\,a^3 + 5\,ab^2 - b^3$.

7. Collect (add): $6\,x^2 - 5\,h + 14\,x - 7\,x^2 - 18\,x + 7\,h - 9\,x + 15\,x^2$.

8. What must be added to $5\,x + 7\,y - 19\,z$ to produce $7\,x - 7\,y - 15\,z$?

A Basket-ball Game

9. The length of a basket-ball court is 20 feet more than its width. If the perimeter of the court is 80 yards, find the width and the length of the court.

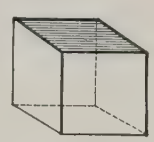
10. Write a formula for the number of square feet (A) in a rectangle l inches long and w inches wide.

11. What is the area in square feet of a rectangle 96 inches long and 36 inches wide? Use the formula obtained in the preceding problem in working this example.

12. The area of a certain square is denoted by x^2. The area of another square 3 feet longer on a side is denoted by $(x + 3)^2$ or by $(x + 3)(x + 3)$. If the area of the smaller square is subtracted from the area of the larger square, the result is 45 square feet. Find a side of each square.

13. Write in algebraic symbols the fact that the length of a circle is equal, approximately, to twenty-two sevenths of the diameter.

14. The edge of a cube is $x + 1$.



(a) Find the sum of all edges.

(b) Find the area of the six faces of the cube.

(c) Find the volume of the cube.

(d) If the edge is doubled, what change takes place in the sum of the edges? In the area? In the volume?

15. The product of two binomials is $x^2 - 11x + 30$. One of them is $x - 6$. How do you find the other? Find it.

16. Multiply $15x + 25y$ by $15x - 25y$ and check.

17. A grocer's record of sales of peaches for a certain year showed that at \$1.25 a basket he sold 10 baskets; at \$1, he sold 30 baskets; at 75 cents, 70 baskets; at 60 cents, 85 baskets; and at 50 cents, 90 baskets. Represent this graphically and from your graph estimate the probable demand when the price is \$1.10 a basket.

18. Find the missing quantity

Product	One Quantity	Missing Quantity
$x^2 - 9x + 20$	$x - 4$	
$x^2 - 13x + 42$	$x - 7$	
$x^2 - 9x - 90$	$x + 6$	

19. Using the formula $r = \frac{I}{Pt}$, find the rate in each of the following:

Interest	Principal	Time	Rate
\$ 120	\$1000	2 yr.	
600	4000	3 yr.	
1440	8000	4 yr.	

20. Make up a formula from the following table:

x	1	2	3	4	5	6
y	3	6	9	12	15	18

21. Draw a graph of the formula $y = 6x$, for $x = 0, 1, 2, 3, 4, 5, 6$. How does y change as x increases?

22. Remove parentheses and combine similar terms:

(a) $9 - 3(4x + 8)$.

(b) $3(x - 1) + 4(x + 2)$.

23. Solve and check:

(a) $\frac{3}{4}x = 24$.

(b) $\frac{1}{5}x + 10 = 50$.

(c) $.5x = 3.5$.

(d) $.7x + 9 = 15.3$.

(e) $20x + 30x - 50 = 150$.

(f) $4(x - 3) + 18 = 34$.

CHAPTER VI

SIMPLE EQUATIONS

We have now learned enough of the different processes of Algebra, so that we can apply them to the solution of problems by means of the equation. Everything we have studied so far has helped lead up to this main topic, so important that, as we have said before, algebra is often called *the science of the equation.*

In our second chapter we were introduced to the four types of equations, and each succeeding chapter has had some work in solving them. We are now ready to take up a more thorough study of equations and to see how a mastery of them simplifies the solution of problems.

53. Equations. — The *sign of equality* is used to state the fact that one algebraic expression is equal to another, as $3x + 2 = x - 6$. Such a statement is called an *equation,* and the expressions on either side of the $=$ sign are called *members of the equation.*

Identities. — There are two different kinds of equations. One is represented by the expression

$$3x + 2x = 5x.$$

Evidently this equation is always true, no matter what value x has. An equation which is always true for every value of the letters concerned is called an *identical equation* or *identity.*

In this kind of equation we may, if we desire, replace the symbol $=$ by the symbol $\equiv$. The latter symbol is usually read "is identical with."

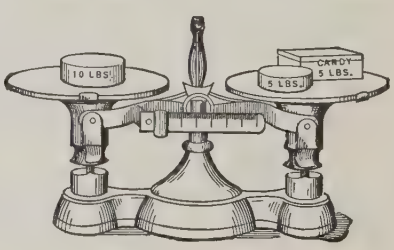
Conditional equations. — The second and more important kind of equation may be illustrated by the expression

$$3x + 2x = 10.$$

Evidently this equation is not true for every value of x. It would not be true, for example, if x were equal to 5, for then $3x + 2x$ would equal 25. An investigation will show that the above equation is true for only one value of x; namely, when $x = 2$. That is, 2 *satisfies* the equation.

An equation which is true for only certain values of the letters involved is called a *conditional equation*. This is the type of equation with which we are chiefly concerned. When we speak of an equation, we mean a conditional equation, unless we specify otherwise.

54. Equations Illustrated by the Balance. — The equation may be illustrated by a pair of balances. The weight in each pan must be kept exactly the same if the balance is maintained. The two members of the equation are like the weights in the two pans of the balance and must be kept exactly equal.



In the diagram there is a 10-pound weight in one pan, and a box of candy and a 5-pound weight in the other. What is the weight of the candy? Then, if $x + 5 = 10$, what does x equal?

In another case the scales balance when there are two equal boxes in one pan and 4 pounds in weights in the other. What does one box weigh? Then, if $2x = 4$, what does x equal?

1. If the scales balance, will they still balance if 10 pounds are added to each pan?

2. Will they balance if 3 pounds are taken from each pan?

3. If the weight in one pan is doubled, what must be done to the weight in the other pan to keep the scales balanced?

4. If a bag of sugar in one pan is balanced by weights in the other, what must be done to the weights to keep the balance, if the sugar is replaced by a bag only half as heavy?

(1) If, then, we add a number to one member of an equation, what must we do to the other member in order to keep the equation true?

(2) If we subtract a number from one member of the equation, what must we do to the other member?

(3) If one member of the equation is multiplied by some number, what must be done to the other member?

(4) If one member of the equation is divided by some number, what must be done to the other member?

55. Axioms. — The principles developed in the preceding paragraphs are called *axioms* and may be stated thus:

1. *If the same quantity be added to both members of an equation, the equation remains true.*

2. *If the same quantity be subtracted from both members of an equation, the equation remains true.*

3. *If both members of the equation be multiplied by the same quantity, the equation remains true.*

4. *If both members of an equation be divided by the same quantity (except* 0*), the equation remains true.*

These axioms are very important principles, and should be carefully memorized by the pupil. They will be referred to hereafter as the axioms of addition, subtraction, multiplication, and division, respectively.

56. Root. — The *root of the equation* is the value of the letter concerned, which makes the equation true, or as we say, which satisfies the equation. Some equations, as we shall see later, have more than one root.

57. The Unknown Quantity. — The *unknown quantity* is the literal quantity in an equation whose value we are to find. It is customary to use the last letters of the alphabet, such as x or y, to represent unknown quantities.

58. The Solution. — To *solve* an equation means to find the value of the unknown quantity, or in other words to find the root which satisfies the equation. The process, when complete, is called the *solution of the equation.*

The method of solving equations by the use of axioms may be made clear by the following

Illustrative examples:

I.

$$6x = 12$$

$x = 2$ Axiom of division.

II. Solve the equation $3x + 2 = 14$.

Solution:

$3x + 2 = 14$

$2 = 2$ Subtract 2 from each member.

$3x = 12$ Axiom of subtraction.

$x = 4$ (root). Axiom of division.

III. Solve the equation $6x - 5 = x + 20$.

Solution:

$6x - 5 = x + 20$

$-x = -x$ Add $-x$ to both members.

$5x - 5 = 20$ Axiom of addition.

$+5 = +5$ Add $+5$ to both members.

$5x = 25$ Axiom of addition.

$x = 5$ (root). Axiom of division.

Check: $6 \cdot 5 - 5 = 5 + 20$

$30 - 5 = 25$

$25 = 25.$

It should be noted that the purpose of these additions is to put the equation in such form that only terms containing the unknown quantity are in the left member while only known quantities are in the right member. Other terms are to be removed: if negative, by addition; if positive, by subtraction.

The additions and subtractions necessary in applying the first and second axioms may soon be done mentally in simple cases.

Checks. — The solution of an equation is checked by substituting the value found, in place of the unknown quantity in the original equation. If the number substituted is a root, the equation will become an identity. In the third illustrative example in section 58 we solved the equation $6x - 5 = x + 20$ and obtained as an answer $x = 5$. To check the solution, we substitute in the place of x in the given equation and obtain $25 = 25$.

Since this is an identity, 5 is a root of the equation.

We are now prepared to state a rule for the solution of simple equations.

Rule. — (1) *By means of the axioms of addition and subtraction remove from the right member of the equation every term containing the unknown quantity, and from the left member every term not containing the unknown quantity.*

(2) *Simplify each member by combining similar terms.*

(3) *Divide both members by the coefficient of the unknown quantity by the use of the axiom of division.*

(4) *The result will be the root of the equation. Check.*

Oral Exercises

Find the value of x in each case:

1. $x + 5 = 10$.
2. $x - 2 = 6$.
3. $2x = 4$.
4. $x - 1 = 12$.
5. $x + 6 = 3$.
6. $2x + 1 = 5$.
7. $3x - 1 = 8$.
8. $4x = 3x + 2$.
9. $2x = x + 6$.
10. $2x + 10 = 30$.
11. $x - 5 = 12$.
12. $2x + 6 = 6$.
13. $3x + 4 = 10$.
14. $4x - 6 = 6$.
15. $6x + 2 = 14$.
16. $2x = 14\frac{4}{5}$.
17. $3x = 21\frac{9}{16}$.
18. $4x = 24\frac{4}{9}$.
19. $5x = 20\frac{10}{13}$.
20. $2x = 12\frac{1}{2} - \frac{1}{6}$.
21. $5x + 5 = 45$.
22. $7x - 16 = 5$.
23. $15x - 5 = 25$.
24. $27x - 3 = 51$.
25. $13\frac{1}{2}x = 27$.
26. $4\frac{1}{3}x = 13$.

59. Changing the Form of an Equation. — There are two convenient ways of changing the form of an equation without affecting its value: (1) by interchanging its members, and (2) by changing all the signs.

1. *Interchanging members.* — The members of an equation may be interchanged by the use of axioms without changing the value of the equation, just as you can turn a balance scale around and still preserve the balance.

Illustrative example:

$40 = 7x + 12$ is the same as $7x + 12 = 40$.
$19 = 5x + 9$ is the same as $5x + 9 = 19$.

2. *Changing signs.* — Sometimes you will find it convenient to change the signs of the terms of an equation by multiplying both members by -1.

Illustrative examples:

$$-6x = 12 \text{ gives } 6x = -12.$$
$$-4x + 3 = 15 \text{ gives } 4x - 3 = -15.$$
$$-5 - 8x = 27 \text{ gives } 5 + 8x = -27.$$

Exercises in Equations

Solve and check:

1. $7x = 2x + 10.$
2. $3x = x + 12.$
3. $10x = 4x - 18.$
4. $6x = 10x + 20.$
5. $5x - 3 = 12.$
6. $8x - 1 = 15.$
7. $5x - 6 = -1.$
8. $4x + 1 = 13.$
9. $9x + 2 = 20.$
10. $6x + 4 = -8.$
11. $16 + 3x = 4.$
12. $4x = 6 + x.$
13. $5 = 2x + 7.$
14. $3x = 15 - 2x.$
15. $12 = 4x - 8.$
16. $8 - x = 6.$
17. $32 - 4x = 8.$
18. $-3x + 6 = 9.$
19. $-9x = 3x + 36.$
20. $20 = -3x + 26.$
21. $7x - 5 = 4x + 1.$
22. $15 - x = -3x + 5.$
23. $4x + 10 = 6x - 2.$
24. $3 - x = 2x - 6.$
25. $10x - 2 = 5x + 8.$
26. $-4 + 2x = 6x + 8.$
27. $6 - 3x = 5x + 2.$
28. $9 - x = 7x - 7.$
29. $3x + 5 = 10 + 2x.$
30. $7 + 3x = 5 - x.$
31. $3y + 6 = 5y + 9.$
32. $6 - 8y = 12 + y.$
33. $2n + 5 = -n - 6.$
34. $7n + 6 = n + 2.$
35. $2z - 4 = 1 - z.$
36. $5z + 1 = 10z + 6.$
37. $3x - 6.2 = x + 8.4.$
38. $1.4x + 5.7 = x - .3.$
39. $4x - (x + 5) = 10.$
40. $2z - (8 + z) = 2.$

60. Solution of Problems. — The usefulness of the equation in algebra may be further shown by the ease with which it enables us to solve such problems as the following:

(a) Five times John's age is 50 years. How old is John?

Let us represent John's age by some convenient letter, such as x. Then, if $5 \cdot x = 50$, what is the value of x?

(b) A box of candy and a 10-cent stamp cost a dollar. What was the candy worth?

Then, if $x + 10 = 100$, what is the value of x?

(c) After having taken a dozen oranges out of a crate there were 60 left. How many were in the crate at first?

Then, if $x - 12 = 60$, what is the value of x?

(d) John is twice as old as his sister and the sum of their ages is 27 years. How old is each?

Here we shall represent the age of the younger by x. Then, since the problem states that John is twice as old, his age must be represented by $2x$. The sum of $2x$ and x is $3x$, and this equals 27. Hence, x must equal $\frac{1}{3}$ of 27, or 9, and $2x = 18$. We arrange the work as follows:

Let x = the number of years in the sister's age.

Let $2x$ = the number of years in John's age.

Then, $3x = 27$, the sum of their ages.

Therefore, $x = 9$ years, the sister's age,

and $2x = 18$ years, John's age.

Check: 18 is twice 9. Also $18 + 9 = 27$.

61. Forming Equations. — Sometimes we must write our own equation from the given problem.

Illustrative example:

Four times a certain number decreased by 12 is equal to 20. What is the number?

Solution:

Let this unknown number be denoted by x.

Then, by the problem, $4x - 12 = 20$

Solving,

$$\begin{array}{rrl} & 12 = 12 & \\ \hline 4x & = 32 & \text{Axiom of addition.} \\ x & = 8 & \text{Axiom of division.} \end{array}$$

Check the answer in the problem.

Therefore 8 is the number required, the root of the equation, the value of the unknown quantity.

Problems

Solve:

1. Henry solved a certain number of algebra problems in an hour, but his older brother Frank solved 3 times as many. Both solved 60. How many were solved by each?

(*Hint:* It is often best to let x stand for the smaller of the unknown quantities. Why?)

2. Francis has 4 times as many marbles as George and both together have 90. How many has each?

3. The sum of the edges of a cube is 24 inches. Find the length of each edge.

4. A house and lot are worth $8400. If the house is worth seven times the value of the lot, find the value of each.

5. The part of a bridge pier out of the water is three times the length of the part under water. If the height of the pier above water is 30 feet, what is the length of the part under water?

6. The length of the Panama Canal is about 50 miles. If the length of the rest of the canal is about nine times the length of the Culebra cut, how long is the Culebra cut?

A Kite-making Contest

7. Matthew's kite string is three times as long as Joseph's. If the strings were tied together, they would reach $\frac{1}{8}$ of a mile. How long is each? (1 mile = 5280 feet.)

8. A line 40 inches long is divided into two parts, so that one part is four times the other part. How long is each part?

9. The perimeter (distance around) of a rectangle is 108 inches. It is twice as long as it is wide. Find the length. The width. Draw the figure.

10. The Missouri-Mississippi River is 12 times as long as the Hudson River, and their combined length is 4550 miles. How long is the Hudson River?

11. One number exceeds (is more than) another number by 6, and the sum of the two numbers is 26. Find the numbers.

A Mississippi River Steamer

Hint: Let x = smaller number.
$x + 6$ = larger number.
$2x + 6$ = sum by adding.
26 = sum.
Then $2x + 6 = 26$. Complete.

12. One number exceeds another number by 13 and the sum of the two numbers is 35. What are the two numbers?

13. A baseball team has lost 4 more games than it has won. If the team has played 46 games, how many games has it lost? How many has it won?

14. Divide a line 80 inches long so that one part exceeds the other by 20 inches.

Hint: Let x = smaller part.
$x + 20$ = larger part.
$2x + 20$ = sum of the parts.

15. The sum of two numbers is 57. One exceeds the other by 17. Find the numbers.

16. The sum of two numbers is 39. One is 5 more than the other. Find the numbers.

17. A man walks 9 miles, then travels a certain distance by automobile, and twice as far by train. If the whole trip is 108 miles, how far does he go by automobile? How far by train?

18. If it requires 180 feet of fencing to inclose a rectangular lot 30 feet wide, how long is the lot? Draw the figure.

62. Transposition. — Let us solve the equation

$$3x - 7 = 11.$$

Solution:

$$\begin{aligned} 3x - 7 &= 11 && (1)\\ +7 &= +7 &&\\ \hline 3x &= 11 + 7 && (2)\ \text{By adding 7 to both members of equation.}\\ 3x &= 18 &&\\ x &= 6\ \text{(root)}. && \text{By dividing both members by 3.} \end{aligned}$$

Again, solve $5x = 12 + x$.

Solution:

$$\begin{aligned} 5x &= 12 + x && (1)\\ x &= x &&\\ \hline 5x - x &= 12 && (2)\ \text{By subtracting } x \text{ from both members.}\\ 4x &= 12 &&\\ x &= 3\ \text{(root)}. && \text{By dividing by 4.} \end{aligned}$$

In the first example, comparing equation (2) with equation (1), we note that instead of the -7 which occurs in the left member of (1) we have $+7$ in the right member of (2). In the second example instead of the $+x$ which occurs in the right-hand member of (1) we have $-x$ in the left-hand member of (2). We might give any number of

illustrations of the fact that when any term is removed from either member of an equation by means of the axioms of addition or subtraction it appears in the other member with the opposite sign.

As a short method of applying these axioms we may, therefore, omit any term from either member of an equation provided we write it in the other member with opposite sign.

This is called *transposition*, and the term so treated is said to be *transposed*. Transposition must not be thought of as a process of mathematics. It is merely an apparent process resulting from the application of the axioms of addition or subtraction.

Illustrative example:

Using transposition, solve $7x - 3 = x + 9$.

Solution: $7x - x = 9 + 3$. By transposing the $+x$ and the -3.

$6x = 12$. Collecting similar terms.

$x = 2$ (root). Dividing by 6.

HISTORICAL NOTE

It is to the principles here explained that the science of algebra owes its name. The word *algebra* is derived from the Arabic *al-gebr*. This word may be translated "the restoration." As here used the word had reference to the fact that the broken parts, as in $3x - 2 = x + 5$, may be restored to a whole, as $2x = 7$, by means of the addition and subtraction axioms as applied in transposition.

63. Combining and Eliminating. — Besides transposition there are two other simple short-cuts that help in solving equations.

One is *combining like terms*. It is best to collect like terms in each member of the equation before going on with the solution.

Thus we solve the equation

$$5x + 12 - 8x = 19 - 13x + 3$$

$12 - 3x = 22 - 13x$ by first collecting terms.

$10x = 10$ Then we transpose and combine.

$x = 1$ By division.

The second is *dropping terms*. If we solve the equation

$$3x + 9 - x = 9 - x + 12$$

$$-9 + x = -9 + x$$ by adding $-9 + x$ to both members

we obtain $3x = 12$

or $x = 4$.

It is clear that $3x = 12$ could have been obtained from the original equation by dropping the $+9$ and $-x$ from both sides of the equation because they occur in both members. Thus identical terms found in each member of the equation should be dropped at the beginning of the solution.

Exercises

Solve for x, y, or z, using transposition or any other device. Check.

1. $5x - 1 = 3x + 7$.
2. $10 - x = 5x + 4$.
3. $3x + 5 = 2x - 1$.
4. $2 - 4x = 14 + 2x$.
5. $y - 6 = 5y - 18$.
6. $20x + 4 = 6x + 60$.
7. $-4 + 3y = -10 + y$.
8. $12x - 5 + x = 60$.
9. $5z + 8 = 7z + 28$.
10. $15 - 8 = 3x + 1$.
11. $-4 + x = -5x + 32$.
12. $16x - 4 = 5x + 40$.
13. $19x + 3 = 7x + 27$.
14. $6x - 3 + 2x = x + 18$.
15. $x + 4 + 3x = 36 - 4x$.
16. $28 - 2x = 5x + 4 - 3x$.
17. $5x + 10 - x = 12 - x + 3$.
18. $2x + 15 - 10 = -3x$.

19. $3x + 12 - x = -2x - 18 - x$.

20. $5z - 2z + 3 = 14 - 2z - 1$.

21. $4x + 13 - 2x - 4 = 45$.

22. $27x + 15 + 12x + 3 - 7x = 15 + 6x - 11 + 12x$.

23. $11x + 13x - 4x - 5 = 6x + 7 + 13x - 5$.

24. $19x - 15 + 9x - 19 - 3x = -15 + 9x - 53 - x$.

25. $4x - 2.5 = 4.5 - 3x$.

26. $.6x - 7 = 9 + .4x$.

27. $.03x - 14 = .03 + 16$.

28. $6.5x - 7 = 12.4 + 4.5x$.

29. $5.7x + 12.2 = 4.6x + 36.2 + .1x$.

30. $8.2x + 6 = 3.7x + 8.4 - 1.5x$.

First combine like terms.

31. $9x - 7 - 4x = 10x + 5 - 7x$.

32. $22x - 9 - 6x = 5x + 16 + x$.

33. $7x - 5 + 2x = 3x + 10 + x$.

34. $4x + 9 - 7x = 8x - 19 + 3x$.

35. $19x + 9 - 12x + 6 = 2x + 35 + 3x$.

36. $14x - 1 + 3x + 5 = 7x + 2 - 4x - 26$.

64. Parentheses in Equations. — The use of a parenthesis or of two adjoining parentheses to indicate multiplication in equations is shown by the following

Illustrative examples:

1. Solve $9x - 3(2x - 5) = 21$.

Solution: $9x - 3(2x - 5) = 21$

$9x - 6x + 15 = 21$ Multiplication.

$3x = 6$

$x = 2$. Answer (root).

Check in original equation.

2. Solve $(x-1)(x+2)-(x+1)(x-2)=6$.

Solution:

$(x^2+x-2)-(x^2-x-2)=6$ By replacing each product by a single parenthesis.

$x^2+x-2-x^2+x+2=6$ Removing the parentheses.

$2x=6$ Collecting like terms.

$x=3$ (root). Dividing by 2.

Check: Giving x the value of 3, in the original equation,

$$(3-1)(3+2)-(3+1)(3-2)=6$$
$$2\cdot 5-4\cdot 1=6$$
$$10-4=6$$
$$6=6.$$

Exercises

Solve, and check:

1. $3(x-1)=2x+6$.
2. $5(2-x)=4x+1$.
3. $4R-5(10-R)=4$.
4. $-x+2(3+x)=7$.
5. $5x-3(6-x)=14$.
6. $5(1-3x)=3x-31$.
7. $3(x-1)-2x=5-x$.
8. $16x-4=3(4x+2)+6$.
9. $15=6+3(4-x)$.
10. $5(1-2x)=x-17$.
11. $4(x+3)-15=25$.
12. $5(2x-8)=4(3x-7)$.
13. $5(2x+3)=3(4x-1)$.
14. $x(x-6)=(x-1)(x+2)-12$.
15. $y+7=y^2-(y+1)(y-3)$.
16. $7x-12-2(x-5)=x-14$.
17. $(x-2)(x+3)=x^2+2$.
18. $(H-1)(H-2)=(H+3)(H-4)$.
19. $x(2x+1)-2x(x+3)=30$.
20. $4x^2-2x(2x+3)=8x+6$.
21. $(x-1)^2-(x+3)^2-88=0$.
22. $4(x-2)^2=(2x+3)^2$.

23. $2x^2 - [(x + 1)(x - 1) + x(x - 3)] = 2x + 6$.

24. $(3x - 1)(x - 1) = 3x^2 - 4x + 3x + 4$.

25. Why may we change $4x = 12 + x$ to $3x = 12$?

26. Why have we the right to change $4 + 5x = 14$ to $5x = 14 - 4$?

27. Does $3x + (2x - 4) = 11$ if $x = 1$? If $x = 2$? If $x = 3$? State a value which makes this equation true.

28. Solve $157 - 21(x + 3) = 163 - 15(2x - 5)$ for x.

29. Solve $(x + 1)(x + 2)(x + 6) = x^3 + 9x^2 + 4(7x - 1)$.

30. Find the value of $17 - 3x$ when $x = 0, 1, 2, 3, 4, 5, 6$. How does $17 - 3x$ change as x increases? What is the value of x if $17 - 3x = 8$? If $17 - 3x = -4$?

65. Equations Having Fractional Coefficients. — Equations of this type may be solved very simply as follows.

Illustrative example:

1. Solve $\frac{x}{2} + 5 = 8$.

Solution: $\frac{x}{2} = 3$ Axiom of subtraction.

$x = 6$. Axiom of multiplication.

2. Solve $\frac{x}{2} + \frac{x}{3} = 5$.

Solution: First find the smallest number into which the denominators 2 and 3 will be contained exactly, which is 6.

$6 \cdot \frac{x}{2} + 6 \cdot \frac{x}{3} = 6 \cdot 5$ Multiplying every term by 6.

or $\frac{6x}{2} + \frac{6x}{3} = 30$

$3x + 2x = 30$ Simplifying.

$5x = 30$ Combining like terms.

$x = 6$. Axiom of division.

Exercises

Solve and check each of the following:

1. $\frac{x}{3} + 6 = 10.$
2. $\frac{x}{4} + 2 = 8.$
3. $\frac{x}{5} + 10 = 15.$
4. $\frac{x}{2} - 4 = 6.$
5. $\frac{x}{5} - 6 = 7.$
6. $\frac{x}{7} - 8 = 10.$
7. $\frac{x}{2} + \frac{x}{3} = 10.$
8. $\frac{x}{3} + \frac{x}{5} = 40.$
9. $\frac{x}{2} - \frac{x}{4} = 12.$
10. $\frac{x+6}{2} = 8.$
11. $.5\,x + .2\,x = .7.$
    Hint: Multiply by 10.
12. $.3\,x + .5\,x = 1.6.$
13. $.9\,x + 1.1\,x = 4.$
14. $.03\,x + .04\,x = 1.4.$

66. Literal Equations. — Equations in which some or all of the known quantities are represented by letters are called *literal* equations. They are solved by the same methods as numerical equations.

Illustrative example:

Solve $x - 2\,a = 3\,x + 4\,a$ for the value of x.

Solution:

$$x - 2\,a = 3\,x + 4\,a$$

$$2\,a = 2\,a$$ Add $2\,a$ to each member.

$$x = 3\,x + 6\,a$$

$$-3\,x = -3\,x$$ Add $-3\,x$ to each member.

$$-2\,x = 6\,a$$ Divide both members by -2.

$$x = -3\,a.$$ Answer.

Check:

$$-3\,a - 2\,a = 3(-3\,a) + 4\,a$$ By substituting the answer in the original equation.

$$-5\,a = -5\,a.$$

Exercises

Solve the following literal equations for x, y, or z and check as directed by your teacher:

1. $2x - a = x + 6a$.
2. $8x + 3b = 6x + 9b$.
3. $4a + x = 10a - x$.
4. $6x - 3a = 3x$.
5. $10x + a = 8x + 13a$.
6. $a - 3x = -x + 5a$.
7. $3y - 2a = -y + 2a$.
8. $x - b = 5x + 7b$.
9. $3a - y = 4y - 2a$.
10. $8z + 6m = 6m + z$.
11. $z - 10n = 10n - z$.
12. $4x + m = 3x - 8m$.
13. $y + 10n = 3y - 4n$.
14. $mn + 3x = 5x - 5mn$.
15. $2x - (a + b) = 3(a + b)$.
16. $3x + 2k = x - 16k$.
17. $4x - m = -m + x$.
18. $2x + 2n = 12n - 3x$.
19. $10y - 2a = y + 16a$.
20. $4a - 2x = 14a + 8x$.
21. $b - a = 4x + 5b + 3a$.
22. $2x + m = x + 3m - n$.
23. $4h + k - 3x = x - 3k$.
24. $3a + 1 - x = 3x + 7a - 1$.

67. Formulas as Literal Equations. — The most important kind of literal equation is the formula, which we studied in our first chapter. Formulas are used so much in business and in science that we must study them very thoroughly.

Exercises

Interpret in words the following formulas:

1. $d = rt$. (d = distance, r = rate, t = time)
2. $A = ab$. (For area of a rectangle)
3. $i = prt$. (i = interest, p = principal, r = rate, t = time in years)
4. $a = p + prt$. (a = amount, other letters as in 3)
5. $A = s^2$. (A = area, s = side of square)

6. $p = ns$. (p = perimeter, s = side of polygon, n = no. of sides)

7. $p = s - c$. (p = profit, s = selling price, c = cost)

8. $l = c - s$. (l = loss, etc.)

Exercises

Write a formula to express each of the following relations, choosing suitable letters to represent the quantities.

1. The area of a triangle is equal to one half the product of its base and altitude.

2. The volume of a rectangular box is equal to the product of its length, breadth, and height.

3. The velocity of a falling body in feet per second is equal to 32 times the number of seconds it has been falling.

4. The area of a circle is equal to $\frac{22}{7}$ times the square of the radius.

5. The average of two numbers is equal to half their sum.

6. The rate per cent of profit is equal to the gain divided by the cost.

7. The number of kilometers in a given distance is equal to the number of miles divided by .6213.

68. Evaluation of Formulas, we can now see, is a simple form of solving literal equations for a letter. If the value of every letter but one in a formula is known, the value of the unknown quantity can be computed. Thus in formula 3, on page 163, let $t = 4$ years, $r = .05$, $p = \$820$, and let it be required to obtain the value of i.

Solution:

$$\begin{aligned} i &= prt \\ &= (\$820)(.05)(4) \\ &= \$164. \end{aligned}$$

Exercises

1. In formula 1 on page 163, obtain d, given $r = 6$ miles, $t = 10$ hours; $r = 12$ mi., $t = 10$ hr.; $r = 24$ mi., $t = 20$ hr.

2. In the formula 2, obtain A, given $a = 3\frac{1}{2}$ inches, $b = 8$ inches; $a = 7''$, $b = 8''$; $a = 7''$, $b = 16''$.

3. In formula 3, obtain i, given $p = \$450$, $r = .06$, $t = 10$ years; $p = \$900$, $r = 6\%$, $t = 10$ yr.; $p = \$450$. $r = 3\%$, $t = 10$ yr.; $p = \$450$, $r = 12\%$, $t = 20$ yr.

4. In formula 4, obtain a, given $p = \$640$, $r = .04$, $t = 8$ years; $p = \$640$, $r = 4\%$, $t = 16$ yr.; $p = \$640$, $r = 4\%$, $t = 24$ yr.; $p = \$1280$, $r = 8\%$, $t = 16$ yr.

5. In formula 5, obtain A, given $s = 6\frac{1}{2}$; $s = 13$; $s = 26$.

6. In formula 6, obtain p, given $n = 5$, $s = 8\frac{1}{2}$; $s = 17$; $s = 25\frac{1}{2}$.

7. In formula 7, find p, given $s = 90¢$, $c = 70¢$; $s = 90¢$, $c = 60¢$; $s = 80¢$, $c = 70¢$; $s = 100¢$, $c = 80¢$.

What would be the meaning of a negative value of p in such a problem?

8. In formula 8, find l, given $c = \$1.20$, $s = \$1.10$.

What would be the meaning of a negative value of l in this problem?

9. Solve formula 7 for s. Also solve for c.

10. Using the answers obtained in example 9, find s if $p = \$10$ and $c = \$80$. Find c if $p = \$.25$ and $s = \$1.75$.

11. Solve formula 8 for c and for s.

12. Using the results obtained in example 11, find c if $l = \$2.50$, $s = \$4$. Find s if $l = \$6$ and $c = \$18$.

13. Using the formula $I = prt$, find the interest on p dollars at 4% for 90 days ($\frac{90}{360}$ of a year). On \$1200. On \$2400. On \$5612. State a short way to find interest at 4% for 90 days.

Practice Test with Formulas

Solve for the letter at the right:

1. $C = np$, for p.
2. $P = 3s$, for s.
3. $A = bh$, for b.
4. $d = vt$, for t.
5. $c = dp$, for d.
6. $C = 2\pi r$, for r.
7. $P = fW$, for f.
8. $W = PN$, for N.
9. $I = prt$, for t.
10. $V = lwh$, for l.
11. $3A = 180°$ for A.
12. $E = CR$, for C.
13. $V = \pi r^2 h$ for h.
14. $S = 2\pi rh$, for h.
15. $C = .05n$, for n.
16. $K = Wh$, for W.
17. $S = r + s$, for s.
18. $C = 8p + 11$, for p.
19. $P = a + b + c$, for a.
20. $P = 2l + 2w$, for l.

Algebraic Representation

1. If John is x years of age, how old was he 5 years ago? a years ago? How old will he be 20 years hence? n years hence?

2. If the length of a rectangle is l and its width is w, what is its perimeter (distance around it)?

3. If one side of a square is s inches, what is its perimeter in inches? In feet?

4. If one apple costs c cents, how many cents will 20 apples cost? a apples? Express each of these answers also in dollars.

5. If oranges are n cents a dozen, what is the cost of one orange? Of 5 oranges? Of x oranges?

6. What is the excess of 50 over x?

7. What number is less than 15 by d?

8. How many hours are required for a train running 40 miles per hour to go 200 miles? To go m miles?

AN AIR-PORT

9. If an airplane flies x miles an hour, how far does it fly in 10 hours? In a hours?

10. How far will a man walk in y hours at the rate of 4 miles per hour?

11. What value of x will make $6x$ equal to 54?

12. How far can I walk in x hours at the rate of y miles per hour?

13. A man has d dollars. How many cents has he? How many dimes?

14. A rectangle is y yards long and w feet wide. What is its perimeter in feet?

69. The Solution of Problems. — In solving applied problems we must form our own equation from the conditions given. We must be sure to read the last sentence carefully, as it often suggests what the unknown quantity should represent.

Hints on Solving Problems.

1. Read the problem carefully to find out just what is required.

2. The number about which we know least is usually the one to represent with a letter as the unknown quantity. Use an initial letter, or x or y, for the unknown quantity.

3. Represent all other numbers in terms of the unknown quantity. For instance, if John is twice as old as Mary, and x stands for Mary's age, then $2x$ will stand for John's age.

4. Form your equation.

5. Solve your equation for the unknown value.

6. Find any other required values, such as John's age in (3).

7. Check in the original problem by substituting the root for the unknown quantity.

Exercises

Number-Relation Problems

1. A certain number of pounds of butter at 60 cents a pound can be bought for \$9. How many were there?

2. A certain number of yards of cloth at $12\frac{1}{2}$ cents a yard cost \$200. Find the number.

3. Six times a certain number minus 2, equals five times the number plus 2. Find the number.

4. One number is three times as large as another and their sum is 8.12. Find each number to two decimal places.

5. Five times a number is 10 more than three times the number. Find the number.

6. If 4 times a number increased by 4 equals the number increased by 16, find the number.

7. The sum of three numbers is 81. The second number is twice the first, and the third is 6 more than the second. Find the numbers.

8. A baseball team has played 84 games and has won 16 games more than it has lost. How many games has it lost?

9. In a certain high school there are 212 more girls than boys. The total number of pupils in the school is 1876. How many boys are there? How many girls?

The Start of the Air Mail

10. A boy went to the post office and bought some postal cards, three times as many two-cents stamps, and two airmail stamps for 10¢. He paid 80 cents. How many 1¢ postal cards did he buy?

Consecutive Number Problems

Oral

Consecutive numbers are numbers which follow in succession, as 8, 9, 10, 11, etc. Consecutive odd numbers are odd numbers which follow in succession, as 3, 5, 7, etc. What are consecutive even numbers?

1. What is the difference between two consecutive numbers? Between two consecutive odd numbers? Between two consecutive even numbers?

2. If one number is x, what is the next larger number? The next larger after that? The number next smaller than x?

3. If one even number is n, what is the next larger even number? What is the next even number before n?

4. Write five consecutive numbers of which x is the middle one.

Written

1. The sum of three consecutive numbers is 42. What are the numbers?

2. There are four consecutive odd numbers whose sum is 96. Find them.

3. Find two consecutive even numbers whose sum is 70.

4. Find three consecutive numbers whose sum is 60.

Problems Involving Percentage

1. How many dollars must a man leave his widow so that if invested at 5% it will produce a yearly income of \$1200?

2. I sold a piece of land for \$4800 and made 20% on the cost. Find the cost.

3. How many dollars must a man leave his widow so that if half of it be invested at 4% and half of it be invested at 5%, it will produce an annual income of \$900?

4. A man wishes to invest \$50,000 in bonds, one part at 5% for safety and the balance at 6%. He needs a yearly income of \$2700. How much must he invest at each rate?

Hint: Let x = the number of dollars invested at 5%.
Then $50{,}000 - x$ = the number of dollars invested at 6%.

Now form a table like the following one.

Invested in Dollars	Rate	Interest
x	5%	$.05\,x$
$50000 - x$	6%	$.06(50000 - x)$

Therefore $.05\,x + .06(50000 - x) = 2700$.

Complete the solution.

5. Mr. Brown invested two equal amounts of money, one at 4% and the other at 6%. His yearly interest was $800. How much did he invest at each rate?

6. At what rate of interest must $8000 be invested to produce an annual income that will pay the rent at $40 a month?

7. If $5000 is invested at 7%, how much must be invested at 4% to give a total income of 6% of the total investment?

Hint: Let x = amount invested at 4%.

Investments	Dollars	Rate	Interest
First	5000	7%	$350
Second	x	4%	$.04\,x$
Total	$5000 + x$	6%	$.06(5000 + x)$

Hence, $.06(5000 + x) = 350 + .04\,x$.

Complete.

8. If $20,000 of a widow's money is invested at 4%, how much must she invest at 8% to give her a total annual income of 6% of her total amount invested?

9. How much must a man invest at 5%, if he already has $12,000 invested at 7%, to give him a total annual income of 6% of his total amount invested?

10. If \$8000 is invested at 5%, and \$6000 at 6%, how much must I invest at 8%, to produce a total annual income of 7% of the total amount invested?

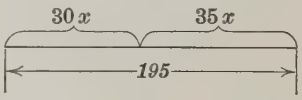
Uniform Motion — Problems

1. Two trains start from the same station and run in opposite directions. One runs at the uniform rate of 30 miles an hour and the other at 35 miles an hour. In how many hours will they be 195 miles apart?

Hint: Let x = the number of hours.

$30\,x$ = distance the first travels.

$35\,x$ = distance the second travels.



$30\,x + 35\,x = 195$.

Complete.

2. Two automobiles start from the same place and travel in opposite directions, the one averaging 25 miles and the other 30 miles an hour. In how many hours will they be 385 miles apart?

3. Two men, A and B, start towards each other at the same time from points 405 miles apart. If they travel 20 and 25 miles an hour respectively, in how many hours will they meet?

4. Jones and Brown start from two points which are 300 miles apart and travel toward each other. The latter travels twice as fast as the former. They meet in 5 hours. Find the rates per hour.

5. A man rides out into the country at a uniform rate of 30 miles an hour. He rests 2 hours and then rides back at 20 miles an hour. He is gone 5 hours. How far did he go?

6. Williams starts out in his auto and travels 20 miles an hour. Four hours later Mr. Speedster starts out from the

same point at 40 miles an hour to overtake Mr. Williams. In how many hours will he overtake him?

Hint: Remember that each will have traveled the same distance when they are together.

7. A motor boat starts out and travels 9 miles an hour. In 3 hours another motor boat traveling 18 miles an hour starts out to overtake the first one. In how many hours will the second boat overtake the first?

Ewing Galloway

OVERTAKING THE TRAIN

8. A train is traveling 30 miles an hour. An automobile starts out from the same place 1 hour later and overtakes the train in 3 hours. What was the rate of the automobile?

9. C and D start from two points 480 miles apart and travel toward each other. They meet in 8 hours. If C travels 6 miles an hour faster than D, find their rates.

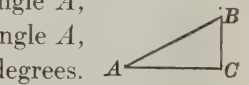
Geometry Problems

1. The length of a rectangle is three times its width and its perimeter is 48 feet. Find its length and its width.

2. A rectangle was formed by placing two equal squares side by side. The perimeter of the rectangle was 18 inches. What was the side of each square?

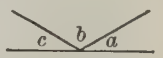
3. 860 rods of wire are used to build a wire fence, five strands high, around a square field. How long is a side of the square?

4. The sum of the three angles of the triangle ABC is 180°. If angle B is twice as large as angle A, and angle C is three times as large as angle A, find the value of each of the angles in degrees.



5. The sum of the angles of a triangle is 180°. If two of these angles are equal and the third angle is three times either of the other angles, how many degrees in each angle?

6. The sum of the angles a, b, and c on the same side of the straight line is 180°. If angle a equals angle c and angle b is four times angle a, how many degrees in each angle?



7. One side of a triangle is twice another. The third side is 8 inches and the perimeter is 26 inches. How long is each side?

8. Two sides of a triangle may be represented by $2n + 1$ and $3n - 2$. If the perimeter is $6n + 5$, what represents the third side? If the length of the perimeter is 65 inches, how long is each side?

9. The second angle of a triangle is 45° more than the smallest angle. The third angle is three times the smallest angle. How many degrees in each angle if the sum of the angles of the triangle is 180°?

10. One acute angle of a right triangle is 5 times as large as the other. How many degrees in each angle?

Hint: Either of the angles, not the right angle, is called an acute angle in the right triangle. Also see example 4.

11. One acute angle of a right triangle is 12° less than the other. How many degrees in each angle?

12. A square has the same area as a rectangle whose length is 8 feet more than a side of the square and whose width is 4 feet less. Find the side of the square.

Mixtures

1. A grocer wishes to mix 60-cent coffee with 40-cent coffee to make 200 pounds to sell at 48 cents a pound. How many pounds of each kind must be used?

Solution: Let n = the number of pounds of 40¢ coffee.
Then $200 - n$ = the number of pounds of 60¢ coffee.
Now form a table like the following.

Kind	Number of Pounds	Value in Cents
40 cent	n	$40\,n$
60 cent	$200 - n$	$60(200 - n)$
48 cent	200	$200 \cdot 48$

Hence, $40\,n + 60(200 - n) = 9600$.
Complete.

2. A grocer wishes to mix 80¢ tea with 50¢ tea to make a mixture of 60 pounds to sell at 60¢ a pound. How many pounds of each kind should he use?

3. A merchant wishes to mix nuts selling at 40¢ a pound with almonds valued at 28¢ a pound. How many pounds of each should he use to obtain a mixture of 120 pounds selling at 35 cents a pound?

4. The radiator of an automobile already contains 6 quarts of a 10% solution of alcohol. How much alcohol must I add to make a mixture of 20% alcohol?

Hint: If x quarts are added, remember that your new mixture will contain $x + 6$ quarts.

Therefore $.20(x + 6) = .6 + x$. Complete.

5. How much alcohol must be added in example 4 to make a mixture containing 25% of alcohol?

6. How many quarts of milk containing 3.5% butter fat and how many quarts of cream containing 30% butter fat must be mixed to make 40 quarts of cream containing 20% butter fat?

Coin Problems

1. 20 coins consisting of dimes and nickels are worth \$1.65. How many nickels are there?

Solution: Let n = the number of nickels.

Then $20 - n$ = the number of dimes.

Now form a table.

Coins	Number	Value in Cents
Nickels	n	$5n$
Dimes	$20 - n$	$10(20 - n)$

Therefore $5n + 10(20 - n) = 165$ (\$1.65 = 165 cents). Complete.

2. 30 coins consisting of nickels and dimes are worth \$2.40. How many of each kind are there?

3. A pocketbook contained 12 coins, all nickels and dimes. Their value was \$1.05. How many nickels were there?

4. Eleven coins consisting of quarters and nickels are worth \$1.95. How many quarters are there?

5. A sum of money amounting to $2.15 consists of dimes and nickels. There are 5 more dimes than nickels. How many coins of each kind are there?

6. A sum of money amounting to $7.50 consists of nickels, dimes, and quarters. There are twice as many dimes as nickels and 10 more quarters than nickels. How many coins of each kind are there?

THE FOOTBALL GAME

7. There were 770 tickets sold for a football game. The pupil tickets cost a quarter and the tickets bought at the gate were a half-dollar each. $228 was taken in. How many of each kind of ticket were sold? What was the average price per ticket?

8. 1003 tickets were sold at a large movie house on a Saturday afternoon, tofaling $213.10. The tickets were a quarter for an adult and a dime for a child. How many of each were sold?

Age Problems

1. A man is 3 times as old as his son. The sum of their ages is 48 years. How old is the son?

2. A man is 3 times as old as his son. Four years ago the sum of their ages was 60 years. How old is the son?

Solution: Let x = son's age now.
Then $3x$ = father's age now.
Now form a table.

	Age Now	Age Four Years Ago
Son	x	$x - 4$
Father	$3x$	$3x - 4$

Therefore $x - 4 + 3x - 4 = 60$.
Complete.

3. A is now 4 times as old as B. Five years ago the sum of their ages was 45 years. How old is each?

4. A is now 4 times as old as B. In 12 years A will be twice as old as B. How old is each?

5. A is now five times as old as B. In 10 years A will be three times as old as B. How old is each?

6. John's age is 2 years more than twice Mary's age, and the sum of their ages is 32 years. How old is each?

Solution: Let x = Mary's age in years.
Then, $2x + 2$ = John's age in years.
And $3x + 2$ = the sum of their ages in years.
But 32 = the sum of their ages in years.
Therefore, $3x + 2 = 32$, etc.

7. John is 5 years older than Winifred. Six years from now the sum of their ages will be 37 years. How old is each now?

8. A father is 5 years more than 4 times his son's age. In 5 years he will be 3 times his son's age then. How old is the son now?

Miscellaneous Problems

1. Kenneth solved a certain number of problems and Harold solved 2 more than twice as many. Together they solved 38. How many did each solve?

2. If 2 is added to a certain number, the result is the same as would be obtained if twice that number were subtracted from 32. What is the number?

3. Harry wished, one summer, to earn enough money for his next year's expenses in college, which would amount to $860. His father said "For every dollar that you earn I will give you four dollars." How much must Harry earn?

4. The number of dollars Frank earned in the summer vacation was 30 less than three times the number which Fred earned. Both together earned $130. How much did each earn?

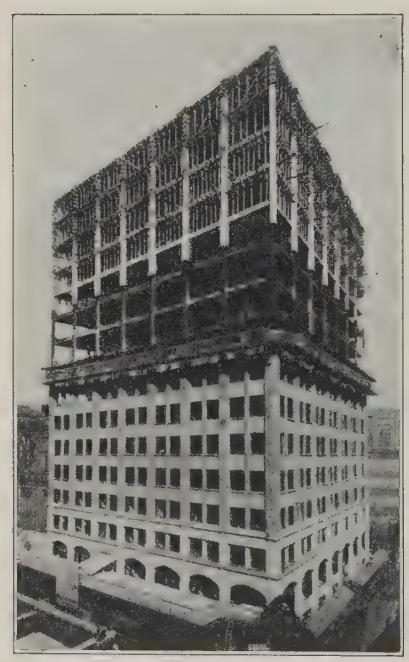
5. Separate 144 into two parts such that one part shall be 28 more than the other part.

6. A steel rod 50 feet long is to be cut into two pieces such that one piece shall be 12 feet longer than the other piece. Find the length of each piece.

7. If one of two partners in a business invests twice as much as the other and they make a profit of $6000 during the year, what should each partner receive?

8. Divide an estate of $3500 among three sons so that the second son gets $500 more than the youngest, and the eldest twice as much as the youngest. How much does each get?

9. A certain kind of concrete contains twice as much sand as cement and 5 times as much gravel as cement. How many cubic feet of each of these materials will there be in 2000 cubic yards of the concrete?



Concrete as Building Material

10. Find each man's share of the profits of a business if A receives twice as much as B, and B receives twice as much as C and the profits for the year are $35,000.

11. A's city tax was $43 more than his county tax. His income tax was $8. His entire tax amounted to $103. How much was his county tax?

12. Seven boys and girls went into an ice cream parlor. Each boy got an ice cream soda at 15¢ and each girl took a chocolate nut sundae at 20¢. The bill was $1.20. How many boys were there? Girls?

13. A man made 40 quarts of lemonade which contained 20% lemon. How much water would he have to add to reduce it to 10% lemon?

14. Mr. Merton left $48,000 to his wife, son, and daughter, in the ratio of 6 to 4 to 2. How much did each receive?

15. The width of a field is to its length as 1 is to 6. If its perimeter is 140 rods, what is the length? The width?

Cumulative Review

1. A line 76 feet long is to be divided into two parts one of which shall be 3 times as long as the other part. How long is each part?

2. First remove the parentheses, and then combine similar terms and find the value of your result by substituting 2 for x and 6 for y in the following:

$$4x - \{2x + 3y - (2y + 4x) - 6\} + 2x.$$

3. Add:

$$\begin{array}{r} -4x + 6y - 3z \\ +3x - 7y + 5z \\ \underline{+7x + \ \ y - 2z} \end{array}$$

Check, using $x = 2$, $y = 3$, $z = 4$.

4. Find the value of $8 - 12.4$.

5. Define and give an example of: (a) An identical equation. (b) A conditional equation.

6. If $x = 3$, $y = 2$, $h = 4$, and $k = 5$, find the value of $(x + y)(x - y) + h(h + k)$ in the shortest way.

Hint:

$$(x + y)(x - y) + h(h + k) = (3 + 2)(3 - 2) + 4(4 + 5)$$
$$= 5 \cdot 1 + 4 \cdot 9, \text{ etc.}$$

7. Apply the values given in example 6 to the following and evaluate: $(3x - y)(2x + y) - (h + k)(4h - 3k)$.

8. Arrange in descending powers of h and multiply:

$7h - 8 - 3h^2$ by $-3 + 2h$. Check with $h = 2$.

9. Write a formula for the total outside surface of a solid similar to a crayon box if the solid is l inches long, w inches wide, and h inches high.

10. Apply your formula of example 9 to such a solid 16 inches long, 8 inches wide, and 4 inches high.

11. Solve $3(x - 2) + 10 = 4(x - 1)$.

12. Is $5x + 6x + 11x = 66$ an identical or conditional equation? Why?

13. Using the formula $r = \frac{d}{t}$, find r if $d = 400$ and $t = 5$. $t = 10$. $t = 20$. How does r change if d remains fixed and t is doubled?

14. Graph the formula $d = 20t$ for $t = 1, 2, 3, 4, 5$. Discuss the changes as t increases.

15. Graph $d = 60t$ and discuss.

16. The following table shows the rates charged by post offices for money orders of various amounts:

Amount of Order	Cost of Sending
$ 2.50	5 cents
5.00	7 cents
10.00	10 cents
20.00	12 cents
40.00	15 cents

(a) Construct a broken-line graph showing the relation between the cost of sending and the amount of the order sent.

(b) State which is changing more rapidly, the amount of the order or the cost of sending it.

17. A pupil's average standing in algebra during each of the first six weeks of the school year is given in the following table:

1st week	86	4th week	88
2nd week	75	5th week	90
3d week	80	6th week	92

Represent this information by means of a bar graph.

PART II

THROUGH QUADRATIC EQUATIONS

CHAPTER VII

PRODUCTS AND FACTORS

In Part I we studied various subjects in algebra, leading up to the solution of problems by means of simple equations. In Part II we shall take up more complicated problems, some of which can be solved only by a different type of equation, called a *quadratic* equation.

The simplest solution of some quadratic equations is by the process of factoring. For this reason our work in Part II begins with the subject of factors.

70. Factoring. — When two or more quantities are multiplied together to form a product, each is called a *factor* of that product. For example, a and b are factors of ab; x and $x - y$ are factors of $x(x - y)$; and 3 and 5 are factors of 15.

A *prime* quantity is a quantity which has no factors except itself and one. Thus, 17 is a prime number and $x + y$ is a prime expression. When a factor can be divided only by itself and one, it is called a *prime factor*. As an illustration we note that 6 and 4 are factors of 24. They are *not* prime factors, however, since 6 and 4 are not prime numbers. The prime factors of 24 are 2, 2, 2, and 3.

Factoring is the process of resolving a quantity into its prime factors. It finds frequent application in algebra. We shall understand factoring more readily if we consider in connection with it certain special products and note how

they are formed. Thus if we know how to form a product mentally, we shall readily recognize the product when we see it and know the factors of which it is composed.

71. Product of a Polynomial by a Monomial. — This case has already been considered in section 46 and the rule is there stated. To refresh our memories we shall work several examples, using horizontal multiplication and writing the results mentally.

Illustrative example I:

$$4(x + 3) = 4x + 12.$$

Illustrative example II:

$$2x^2(y^3 - 6y^2 + 5y + 1) = 2x^2y^3 - 12x^2y^2 + 10x^2y + 2x^2.$$

Exercises

Multiply, as in the above illustrative examples:

1. $5(x + 6)$.
2. $7(x - 3)$.
3. $3x(x + 5)$.
4. $2x(x - 6)$.
5. $-5(x + 7)$.
6. $-2(2x + 5)$.
7. $3(3x + 2)$.
8. $2(x + y + z)$.
9. $3x(x + 5y + z)$.
10. $xy(x + y)$.
11. $a^2b(3a + 2b - ab)$.
12. $3m(m^2 + mn - mn^2)$.
13. $2x(x^3 + x^2 - 2x + 1)$.
14. $3x(x^4 + 2x^3 - 3x - 2)$.
15. $3s^2(s - m + x - 3)$.
16. $x(3 + 4x - 5x^2)$.
17. $3(2x^2 - 3xy - 5y^2)$.
18. $2m^2n(m^2 - 3mn + 5n^2)$.
19. $6x^2y^3(2x + 3y + 2xy - y^2)$.
20. $y^3(y^3 - 2y^2 + 4y + 1)$.
21. $a^4(4a^2 - 5a + 1)$.
22. $abc(bc + ca + ab)$.
23. $-5ab(3a^2 - 2ab + 7b^2)$.
24. $-6x^3(4 - 3x^2 + 3x^3 - 4x^4)$.

Optional

Simplify:

25. $c(a + b) - c(a - b)$.

26. $7\,a(b - c) - 2\,b(a - c)$.

27. $\frac{1}{2}(x - 2\,y) - \frac{3}{4}(y - 2\,x)$.

28. $2[3\,xy - 4\,x(z - 2\,y)]$.

29. $A^9(A^3 + A^2 - A - 1)$.

30. $.5\,x(.25\,x^2 + .75\,x^3 - .6\,x^4)$.

72. Monomial Factors. — Since all these products are formed by multiplying each term of the polynomial by the monomial, it is clear that in every case the monomial multiplier is contained in each term of the product.

Illustrative examples:

1. $4\,x + 4 = 4(x + 1)$.

2. Find the factors of $3\,x^3 + 6\,x^2y + 9\,x^2z + 3\,x^2$.

A study of the terms of this polynomial shows that the monomial expression $3\,x^2$ is contained in each. $3\,x^2$, then, is one of the factors of the polynomial. The other factor may be obtained by dividing the given expression by the monomial factor already found. Thus we obtain the factors

$$3\,x^2(x + 2\,y + 3\,z + 1).$$

Note that $3\,x^2 = 3 \cdot x \cdot x$.

It is not customary, however, to change a monomial factor into its prime factors, since $3\,x^2$ by its form indicates the product of 3 and x and x. The expression may therefore be considered as completely factored if left in the form given above.

We may now state the following

Rule. — *To find the factors of an expression which is the product of a monomial and a polynomial*

(1) *Study the terms of the given expression to obtain the largest monomial common to them all.*

(2) *Divide the expression mentally by this monomial to obtain the polynomial factor.*

(3) *Indicate the product of the factors found.*

Exercises

Find the factors of each of the following:

1. $3x + 3y$.
2. $4h - 8k$.
3. $5x - 15y$.
4. $8x + 24y$.
5. $3x + 9y$.
6. $xy + xz$.
7. $26x + 39y$.
8. $25 - 50x$.
9. $P + Prt$.
10. $\pi R^2 - \pi r^2$.
11. $xyz + xyzw$.
12. $4k - 4k^2$.
13. $3x^3 + 6x^4$.
14. $3x^2 + 6x^2y$.
15. $4a^2bc + 8abc^2$.
16. $\frac{1}{2}ax + \frac{1}{2}ay$.
17. $\frac{1}{4}ak + \frac{1}{4}ah$.
18. $.6x - 1.2y$.
19. $27x^4 - 18x^2$.
20. $36x^4 - 24x$.
21. $2\pi r^2 + 2\pi rh$.
22. $72x^3 - 54x$.
23. $\frac{1}{3}\pi r^2h + \frac{1}{3}\pi R^2h$.
24. $.2y^3 - .4by^2$.
25. $\pi r^2 + \pi rl$.
26. $6x^3 - 12x^2 + 24x$.
27. $8H^2 + 32HK + 16$.
28. $xy + xz + xw$.
29. $x^4 + x^2 + x$.
30. $2x^2 + 4bx + 6x$.
31. $10a^4 + 25a^3 - 45a^2$.
32. $-x^3 + x^2 - x$.
33. $23 \cdot 60 + 23 \cdot 40 - 23 \cdot 75$.
34. $13 \cdot 48 + 13 \cdot 36 + 13 \cdot 49$.
35. $635 \cdot 724 - 635 \cdot 721$.
36. $150 \cdot 63 + 100 \cdot 63 - 200 \cdot 63$.
37. $3.1416 \times 8.432 - 3.1416 \times 6.432$.
38. 75 lb. @ 25¢ − 50 lb. @ 25¢.
39. $4ab - 12abc + 16a^2b^2c^2$.
40. $14x - 21a + 49bx$.
41. $2a^2b - 2ab^2 - 2a^2b^2$.
42. $3s^2 + 6sy + 15s^2y^2$.
43. $-m^2n - mn^2 - 3m^2n^2$.

Optional

44. $5\,x^n + 10\,y^n$.

45. $6\,x^m + 12\,y^m$.

46. $20\,x^{2n} + 45\,x^n$.

47. $3\,hx^{2n} - 6\,hx^{4n}$.

48. $2\,hx^{2a} - 6\,hx^a + 8h$.

49. $\frac{1}{3}\,hK + \frac{1}{3}\,aK + \frac{1}{3}\,bK$.

50. $\frac{3}{4}\,a^2 + \frac{3}{4}\,b^2 - \frac{3}{4}\,c^2$.

51. $.4\,x + .4\,y - .4\,z$.

52. $.3\,a - .3\,b + .3\,c$.

53. $x^4 + x^3 - x^2 + x$.

54. $24\,x^4 - 36\,x^3 + 18\,x^2 - 3\,x$.

55. $3\,x^2 + 9\,x^3 - 12\,x^4 + 15\,x$.

56. $2\,xyz^2 - 6\,x^2yz + 10\,xy^2z$.

57. $ax^2y - 5\,a^2x^2 - 2\,a^3x^4y^2$.

58. $8\,a^2b^2c^2 + 4\,ab^2c^4 - 3\,ab^3c + 5\,a^5b^4c$.

59. $15\,m^2nx - 20\,mnx^2 - 60\,m^3n^3x^4$.

60. $16\,s^2t^2 - 64\,st^3 + 48\,s^3t^2 + 96\,s^3t^5$.

61. $121\,x^2y^3 + 44\,x^3y^2 + 110\,xy^4 - 33\,x^3y^5$.

62. $x(a + b) + y(a + b)$.

63. $5\,x(a - b) - 3\,y(a - b)$.

64. The area of a certain rectangle is $3\,x^2 + 3\,xy$. If the factors of this expression denote the sides of the rectangle, what are the sides?

65. Find the dimensions of a rectangle whose area is $6\,x^2 + 9\,xy$. Whose area is $15\,xy + 10\,y^2$.

73. Product of the Sum of Two Terms by Their Difference. — If we multiply $3\,x + 2\,y$ by $3\,x - 2\,y$, we obtain $(3\,x + 2\,y)(3\,x - 2\,y) = 9\,x^2 - 4\,y^2$.

$9\,x^2$ is the square of $3\,x$, $4\,y^2$ is the square of $2\,y$, and the sign between these terms is negative. We might, therefore, appropriately describe this product as the *difference of the squares* of $3\,x$ and $2\,y$. We have, then, the following

Rule. — *If the sum of two terms is multiplied by the difference of the same two terms, the product is the difference of their squares.*

In applying this rule it is to be noted that the difference of the squares occurs in the same order as the difference of the terms.

Exercises

Write by the preceding rule the products of:

1. $(x + 2)(x - 2)$.
2. $(x + 4)(x - 4)$.
3. $(x - 5)(x + 5)$.
4. $(x + 8)(x - 8)$.
5. $(x + 10)(x - 10)$.
6. $(x + 12)(x - 12)$.
7. $(x + 1)(x - 1)$.
8. $(x + 7)(x - 7)$.
9. $(x + 11)(x - 11)$.
10. $(x + 20)(x - 20)$.
11. $(x + 3)(x - 3)$.
12. $(a + 6)(a - 6)$.
13. $(n + 9)(n - 9)$.
14. $(x^2 - 3)(x^2 + 3)$.
15. $(x^2 + 12)(x^2 - 12)$.
16. $(x^4 - 4)(x^4 + 4)$.
17. $(9 + x^3)(9 - x^3)$.
18. $(2a + 1)(2a - 1)$.
19. $(3a - 2)(3a + 2)$.
20. $(4x - 3)(4x + 3)$.
21. $(5y - 4)(5y + 4)$.
22. $(6s - 7)(6s + 7)$.
23. $(2 - 5x)(2 + 5x)$.
24. $(5 + 3a)(5 - 3a)$.
25. $(10 + 7m)(10 - 7m)$.
26. $(4H + 5K)(4H - 5K)$.
27. $(11s - 3b)(11s + 3b)$.
28. $(x - 2mn)(x + 2mn)$.
29. $(4a + 7b^2)(4a - 7b^2)$.
30. $(2a^2 + 1)(2a^2 - 1)$.
31. $(1 + 2ab^2)(1 - 2ab^2)$.
32. $(3x^2 - 2y)(3x^2 + 2y)$.
33. $(5x^2y + ab)(5x^2y - ab)$.
34. $(\frac{1}{2}a + \frac{1}{3}b)(\frac{1}{2}a - \frac{1}{3}b)$.
35. $(\frac{3}{4}x - \frac{1}{2})(\frac{3}{4}x + \frac{1}{2})$.
36. $(\frac{1}{5}x^2 + \frac{3}{2})(\frac{1}{5}x^2 - \frac{3}{2})$.
37. $(\frac{2}{3}x^2 - 4)(4 + \frac{2}{3}x^2)$.
38. $(.5x + .8y)(.5x - .8y)$.
39. $(.3a^2 - .4b^2)(.3a^2 + .4b^2)$.
40. $102 \cdot 98$.

Solution: $102 \cdot 98 = (100 + 2)(100 - 2)$
$= 100^2 - 2^2 = 10{,}000 - 4 = 9996$.

41. $52 \cdot 48$. **43.** $61 \cdot 59$. **45.** $77 \cdot 63$. **47.** $58 \cdot 62$.

42. $35 \cdot 25$. **44.** $27 \cdot 33$. **46.** $42 \cdot 38$. **48.** $46 \cdot 54$.

Find the cost of

49. 65 golf balls at 75¢ each.

50. 42 books at 38c each.

51. 24 lb. of coffee at 36¢ a pound.

52. 57 dozen eggs at 63¢ a dozen.

53. Find the area of a rectangle 52 feet by 48 feet.

Optional

54. $(.2\,xy + .5)(.2\,xy - .5) = ?$

55. $(.6\,H + .05\,K)(.6\,H - .05\,K) = ?$

56. $(a^n + 3)(a^n - 3) = ?$

57. $(a^x + b^2)(a^x - b^2) = ?$

58. $(3\,e^{2x} + 1)(3\,e^{2x} - 1) = ?$

59. $(.5\,xy + .6)(.5\,xy - .6) = ?$

60. $(7\,x^3y + 3\,z)(7\,x^3y - 3\,z) = ?$

61. $(a^2 - b^2) \div (a + b) = ?$

62. $(25 - x^2) \div (5 - x) = ?$

What is a binomial divisor of each of the following?

63. $a^2 - b^2$. **64.** $x^4 - 9$. **65.** $9\,x^2 - 16$. **66.** $36 - 25\,y^2$.

State the two binomials whose product is:

67. $x^2 - y^2$. **68.** $H^2 - 36$. **69.** $64 - 4\,y^2$.

70. (The square of a plus the square of b) multiplied by (the square of a minus the square of b) equals what?

71. (x cube minus y square) times (x cube plus y square) = ?

72. (x fifth minus y fourth) times (x fifth plus y fourth) = ?

74. Factors of the Difference of Squares. — A product of the form obtained in the preceding exercise is described as the *difference of squares*. $36\,y^2 - 25$ is such a product.

We know at once from the preceding section that its factors are $6y + 5$ and $6y - 5$. Since $36y^2$ is the square of $6y$, $6y$ is called the square root of $36y^2$.

In the same way $9n^2 - 4m^2 = (3n + 2m)(3n - 2m)$. Since $9n^2$ is the square of $3n$, we call $3n$ the *square root* of $9n^2$, and $2m$ the square root of $4m^2$.

Also, $5^2 - 2^2 = (5 + 2)(5 - 2)$.

$$n^4 - m^4 = (n^2 + m^2)(n^2 - m^2)$$
$$= (n^2 + m^2)(n + m)(n - m).$$

Rule. — *The factors of a binomial which is the difference of two squares are the sum and the difference of their square roots.*

Exercises

Factor each of the following and check:

1. $x^2 - 4$.
2. $x^2 - 9$.
3. $x^2 - 16$.
4. $x^2 - 36$.
5. $x^2 - 100$.
6. $x^2 - 81$.
7. $49 - x^2$.
8. $144 - x^2$.
9. $x^2 - y^2$.
10. $s^2 - 25$.
11. $1 - 16x^2$.
12. $1 - 49y^2$.
13. $16x^2 - 9$.
14. $25y^2 - 16z^2$.
15. $4s^2 - 49t^2$.
16. $49m^2 - 121n^2$.
17. $81x^2 - 25a^2$.
18. $144l^2 - 121k^2$.
19. $25 - 16n^2$.
20. $100 - 49a^2$.
21. $36a^2b^2 - 25c^2$.
22. $x^2 - \frac{1}{9}$.
23. $x^2 - \frac{1}{16}$.
24. $\frac{1}{4}x^2y^4 - \frac{1}{25}z^6$.
25. $\frac{9}{4}a^2 - \frac{4}{9}b^2$.
26. $16 - 9t^2s^4$.
27. $64a^2b^2 - 81c^2$.
28. $s^2 - 16t^2y^2$.
29. $36m^2 - 25s^2t^2y^6$.
30. $a^2b^4 - 9$.
31. $4a^2 - 9b^2$.
32. $16x^2 - 25y^2$.
33. $x^4 - y^4$.
34. $y^4 - 1$.
35. $n^8 - m^8$.
36. $1 - .36x^2$.
37. $.04 - .25x^2$.
38. $.09H^2 - .16K^2$.
39. $25h^2 - \frac{1}{4}k^2$.

Find mentally the values of $15^2 - 5^2$.

Solution: $15^2 - 5^2 = (15 + 5)(15 - 5)$
$= 20 \cdot 10$, or 200.

Find mentally the values of the following:

40. $18^2 - 12^2$.
41. $39^2 - 31^2$.
42. $77^2 - 73^2$.
43. $25^2 - 15^2$.
44. $64^2 - 36^2$.
45. $90^2 - 10^2$.
46. $700^2 - 300^2$.
47. $432^2 - 431^2$.
48. $566^2 - 565^2$.
49. $245^2 - 243^2$.
50. $3424^2 - 3423^2$.
51. $8.68^2 - 1.32^2$.

52. If $\pi = 3.1416$, $R = 35$, $r = 25$, find the value of $\pi R^2 - \pi r^2$ in a short way by the use of factoring.

Optional

Factor:

53. $.64\,x^2 - .0049\,y^2$.
54. $.16\,x^2 - \frac{1}{16}\,y^2$.
55. $x^{2n} - y^{2m}$.
56. $144\,h^{2n} - k^{4m}$.
57. $\frac{25}{a^2} - \frac{81}{b^2}$.
58. $\frac{16\,a^2}{b^2} - \frac{x^2}{49}$.

You may have to factor an expression of two or more types.

59. $x^3 - 16\,x$.

Hint: This $= x(x^2 - 16)$
$= x(\quad)(\quad)$

60. $5\,x^2 - 45$.
61. $2\,x^2 - 32$.
62. $x^3 - 4\,x$.
63. $a^3 - 16\,a$.
64. $e^3 - 25\,e$.
65. $7\,e^3 - 7\,e$.
66. $\pi R^2 - \pi r^2$.
67. $2\pi R^3 - 8\pi RH^2$.
68. $x^4y^5 - yz^2$.
69. $242\,am^2 - 162\,ax^2y^2$.
70. $x^3y^2 - 64\,x^3$.
71. $11\,x^4 - 44\,x^2$.
72. $abx^2 - 9\,aby^2$.
73. $27\,t^2s - 48\,sy^2$.

Factor $(x + y)^2 - 16$.

Solution: $(x + y)^2 - 16 = [(x + y) + 4][(x + y) - 4]$
$= (x + y + 4)(x + y - 4)$.

74. $(x + y)^2 - 25$.

75. $(a - b)^2 - e^2$.

76. $x^2 - (y + z)^2$.

77. $9(a - b)^2 - 16$.

78. $(a + b)^2 - c^2$.

79. $(x + 1)^2 - 4\,n^2$.

80. $25 - 36(c - d)^2$.

81. $m^2 - (x - y)^2$.

82. $4\,s^2 - 25(a - 2\,b)^2$.

83. $16\,m^2(a - b)^2 - 49\,c^2$.

84. $9(x - y)^2 - 25$.

85. $(x - y)^2 - (a + b)^2$.

86. $(2\,x - 1)^2 - (3\,m - n)^2$

75. Products of Binomials Having Similar Terms. — Let us consider the product of the binomials $2\,x - 2\,y$ and $3\,x + 4\,y$, in which the terms of one are respectively similar to those of the other. If the multiplication is performed in the usual way, it is seen that the first term of the product, $3\,x^2$, is obtained by multiplying the first terms of the binomials; namely, $3\,x$ and x. In like manner the last term of the product, $-8\,y^2$, is obtained by multiplying the last terms of the binomials; namely, $4\,y$ and $-2\,y$. The middle term of the product, $-2\,xy$, is obtained by adding to the product of x and $4\,y$ the product of $3\,x$ and $-2\,y$. These are called "cross products" and are indicated in the example by the cross lines.

$$\begin{array}{r} 3x + 4y \\ x - 2y \\ \hline 3x^2 + 4xy \\ \quad - 6xy - 8y^2 \\ \hline 3x^2 - 2xy - 8y^2 \end{array}$$

The form $(3\,x + 4\,y)(x - 2\,y) = 3\,x^2 - 2\,xy - 8\,y^2$ is more convenient for mental work. Here the terms which are multiplied to form the cross products are connected by the curved lines.

Formulate for yourself a rule for the mental multiplication of binomials of the type above.

Exercises

Mentally write the product of:

1. $(x + 2)(x + 7)$.
2. $(x + 3)(x + 4)$.
3. $(x + 5)(x + 6)$.
4. $(x + 8)(x + 3)$.
5. $(x + 1)(x + 9)$.
6. $(x - 3)(x - 4)$.
7. $(x - 5)(x - 2)$.
8. $(a - 6)(a - 6)$.
9. $(K - 8)(K - 2)$.
10. $(a - b)(a - b)$.
11. $(x - 7)(3x - 1)$.
12. $(x - 3)(x + 1)$.
13. $(x - 2a)(x + 5a)$.
14. $(m - 4n)(m + 7n)$.
15. $(m + 5n)(m - 8n)$.
16. $(A + 3B)(A - 6B)$.
17. $(hk + 6)(hk - 9)$.
18. $(xy - 12)(xy + 8)$.
19. $(2x + 3)(3x + 4)$.
20. $(3x + 5)(4x + 7)$.
21. $(5x + 1)(x + 4)$.
22. $(3x + 2)(x - 3)$.
23. $(2x + 5)(x - 5)$.
24. $(2a - 3)(7a + 1)$.
25. $(3y - 1)(y - 1)$.
26. $(4x + 1)(3x - 2)$.
27. $(7y - 1)(7y - 1)$.
28. $(11x + 6)(9x - 3)$.
29. $(7x + 8)(6x - 9)$.
30. $(1 - 5b)(1 + 3b)$.
31. $(1 + 6b)(1 - 5b)$.
32. $(2y^2 + 5)(2y^2 + 3)$.
33. $(12x^2 + 3)(4x^2 - 7)$.
34. $(3x^3 - 4)(5x^3 + 6)$.
35. $(9x^2 - 5)(5x^2 + 7)$.
36. $(7x + 2a)(x - 3a)$.
37. $(2a - 3b)(a + 2b)$.
38. $(2x - a)(2x - 3a)$.
39. $(x - 10y)(5x + 3y)$.
40. $(3 + 2a)(3 + 2a)$.
41. $(2x - 3y)(3x + 2y)$.
42. $(2s + t)(3s + t)$.
43. $(2a - 3x)(5a + 7x)$.
44. $(11x - 3y)(x + 2y)$.
45. $(2a + 5b)(2a + 5b)$.
46. $(x^2 - 30)(x^2 + 20)$.
47. $(x^2 + 50)(x^2 - 40)$.
48. $(9x^2y - 3z)(2x^2y - z)$.
49. $(11y^2 + 9z)(3y^2 - z)$.
50. $(7x^2 + y^2)(5x^2 - y^2)$.

Optional

51. $(8x^2-y^2)(9x^2-2y^2)$.
52. $(7a^3-3b)(5a^3+b)$.
53. $(12x^2-4y)(5x^2+6y)$.
54. $(9x^2-8)(5x^2+6)$.
55. $(\frac{1}{2}a-b)(2a+10b)$.
56. $(\frac{1}{2}x+6)(\frac{1}{3}x+4)$.
57. $(\frac{3}{4}x-8)(\frac{1}{2}x+4)$.
58. $(a-\frac{2}{3})(a-\frac{1}{3})$.
59. $(k+\frac{1}{2})(\frac{3}{4}k+\frac{1}{2})$.
60. $(x-4)(x+\frac{3}{2})$.
61. $(x+\frac{3}{4})(x-\frac{2}{3})$.
62. $(y+.5)(y+.2)$.
63. $(h-.05)(h-.02)$.
64. $(.3y+.2x)(.4y-.5x)$.
65. $(x^a+3)(x^a+6)$.
66. $(x^a-4)(x^a-7)$.
67. $(e^x-3)(e^x+1)$.
68. $(2e^x-3)(e^x+4)$.
69. $(4x^a+1)(x^a-2)$.
70. $(5-e^x)(6+e^x)$.

76. Factors of Quadratic Trinomials. — From the examples in the preceding section, we see that the product of two binomials having similar terms is usually a trinomial. This trinomial always contains the squares of the letters involved, and hence is called a *quadratic trinomial* (from the Latin word *quadratus*, meaning "*squared*").

If it is required to find the factors of a quadratic trinomial, we need only to reverse the method of the last exercise.

Illustrative example:

Find the factors of $2x^2-x-15$.

We know there will be two factors, each a binomial, hence we write the adjacent parentheses ( )( ). We know that the first term, $2x^2$, is the product of the first terms of the binomial factors, hence one must be $2x$, the other x. We therefore have $(2x\ \ )(x\ \ )$.

The last term, -15, is the product of the last terms of the binomial factors and, since it is negative, these terms must have opposite signs. The possibilities are -3 and $+5$, $+3$ and -5, -15 and $+1$, $+15$ and -1.

If we choose the last of these, we have $(2x + 15)(x - 1)$. We cannot be sure that the factors obtained are correct until we have tested the product to see that it gives the correct middle term. Forming the middle term by the method of cross products, as explained in section 75, we obtain $+ 13x$. As this is not the middle term of the given example, these factors are incorrect.

As a second trial we might take $(2x - 3)(x + 5)$. Here again the middle term is found to be incorrect. An interchange of $+ 5$ and $- 3$ gives us $(2x + 5)(x - 3)$ and an investigation of the middle term shows that these are the correct factors.

Several trials may be necessary before correct factors of a quadratic trinomial can be found, but a little experience will soon enable us to eliminate most of the incorrect trials and to find the solution quickly.

Hints: 1. If the last term of the trinomial is $+$, the signs of the last terms of the binomial factors will be alike. They will both be $+$ or both be $-$, according as the sign of the middle term of the trinomial is $+$ or $-$.

2. If the last term of the trimonial is $-$, the last terms of the factors will have unlike signs.

Exercises

Find the factors of

1. $x^2 + 6x + 8$.
2. $x^2 + 9x + 20$.
3. $x^2 + 6x + 9$.
4. $x^2 - 7x + 10$.
5. $y^2 - 6y + 8$.
6. $a^2 - 3a + 2$.
7. $x^2 + 3x - 10$.
8. $x^2 + 4x + 3$.
9. $x^2 + 5x - 6$.
10. $x^2 - 9x + 14$.
11. $x^2 + x - 12$.
12. $x^2 - 3x - 10$.
13. $x^2 - 6x + 8$.
14. $x^2 + 5x - 14$.

15. $x^2 - 8x + 15$.

16. $x^2 + 9x - 22$.

17. $x^2 + x - 56$.

18. $x^2 - 12x + 27$.

19. $x^2 - 8x - 9$.

20. $x^2 + 10x + 9$.

21. $x^2 - 10x + 21$.

22. $a^2 - 13a + 42$.

23. $a^2 - a - 56$.

24. $n^2 - 19n + 88$.

25. $n^2 - 10n - 24$.

26. $3 - 2x - x^2$.

27. $10 + 3y - y^2$.

28. $66 - 17y + y^2$.

29. $1 - 3x - 10x^2$.

30. $1 - 10y + 25y^2$.

31. $y^2 - by - 2b^2$.

32. $2x^2 - x - 3$.

33. $6y^2 - 5y - 4$.

34. $3y^2 - 4y - 15$.

35. $2m^2 + m - 21$.

36. $10 - 13x - 3x^2$.

37. $3R^2 - R - 4$.

38. $4x^2 - x - 5$.

39. $3x^2 - 5x - 12$.

40. $10x^2 + 11x - 6$.

41. $18a^2 - 27a + 4$.

42. $3x^2 - 2x - 1$.

43. $3H^2 + 10H + 7$.

44. $12a^2 + 4a - 1$.

45. $12m^2 - 7m + 1$.

46. $12y^2 - xy - x^2$.

47. $9x^2 - 6xy + y^2$.

48. $2x^2 + 3xy - 5y^2$.

49. $4m^2 + 17mn + 18n^2$.

50. $6m^2 - 31ms + 5s^2$.

Optional

51. $4x^2 - 12xy + 9y^2$.

52. $15a^2 - 26ab - 24b^2$.

53. $2a^2 - 5ab - 12b^2$.

54. $3x^2 - 7xy - 6y^2$.

55. $4x^2 - 4x - 3$.

56. $6a^2 - 13ab + 6b^2$.

57. $x^2 - x + \frac{1}{4}$.

58. $2a^2 + 23a - 12$.

59. $12x^2 - 31x - 15$.

60. $x^{2a} + 8x^a + 15$.

61. $e^{2x} - 2e^x - 63$.

62. $3a^{2n} - 8a^nb - 3b^2$.

63. $(x+y)^2+10(x+y)+21$.

64. $(x+y)^2-11(y+y)+28$.

65. $(x+y)^2+2(x+y)-35$.

66. $2x^2y + xy - 3y$.

Hint: $2x^2y + xy - 3y = y(2x^2 + x - 3)$. Complete.

67. $6x^3 - x^2y - xy^2$.

68. $6a^2 + 10ab - 4b^2$.

69. $40 + 2x - 2x^2$.

70. $2x^2 - 10x - 28$.

71. $24x^2y + 10xy - 6y$.

72. $2x^2z - 12xyz + 18y^2z$.

73. $2m^3n - 19m^2n + 24mn$.

74. $y^3 - 6y^2 + 9y$.

75. $8x^2 + 22x + 12$.

76. $6m^3 + 8m^2 - 8m$.

77. $x^2 + .7x + .12$.

78. $x^2 + 1.3x + .40$.

79. $x^2 - .6x - .27$.

80. $x^2 + .05x + .0006$.

77. The Square of a Binomial. —

Illustrative examples:

I. $(a + b)^2 = (a + b)(a + b)$
$= a^2 + 2ab + b^2$

II. $(a - b)^2 = (a - b)(a - b)$
$= a^2 - 2ab + b^2$.

III. $(5x + 4y)^2 = 25x^2 + 40xy + 16y^2$.

IV. $105^2 = (100 + 5)^2$
$= 10{,}000 + 2 \cdot 500 + 25$
$= 11{,}025$.

V. $98^2 = (100 - 2)^2$
$= 10{,}000 - 2 \cdot 200 + 4$
$= 9604$.

The result then of squaring any binomial may be written at once by the following

Rule.[1] — (1) *The square of the sum of two numbers equals the square of the first plus twice the product of the first by the second, plus the square of the second.*

(2) *The square of the difference of two numbers equals the square of the first, minus twice the product of the first by the second, plus the square of the second.*

[1] This rule is sometimes stated: (1) The square of the sum of two numbers equals the sum of their squares plus twice their product. (2) The square of the difference of two numbers equals the sum of their squares minus twice their product.

Exercises

Square the following binomials by inspection:

1. $(x + y)^2$.
2. $(x + 5)^2$.
3. $(x + 8)^2$.
4. $(x + 12)^2$.
5. $(x + 1)^2$.
6. $(x + 2)^2$.
7. $(x + 3)^2$.
8. $(x + 4)^2$
9. $(x + 6)^2$.
10. $(x + 7)^2$.
11. $(b + c)^2$.
12. $(x - y)^2$.
13. $(x - 5)^2$.
14. $(x - 8)^2$.
15. $(x - 12)^2$.
16. $(x - 1)^2$.
17. $(x - 2)^2$.
18. $(x - 3)^2$.
19. $(x - 4)^2$.
20. $(x - 6)^2$.
21. $(x - 7)^2$.
22. $(b - c)^2$.
23. $(x^2 + 12)^2$.
24. $(x^2 - 11)^2$.
25. $(3x + 5)^2$.
26. $(4x + 3)^2$.
27. $(5x - 2)^2$.
28. $(6x - 1)^2$.
29. $(3x^3 - 4)^2$.
30. $(4x^2 - 5)^2$.
31. $(x - 3y)^2$.
32. $(b + 3a)^2$.
33. $(x + 2y)^2$.
34. $(2x - y)^2$.
35. $(3a + b)^2$.
36. $(2a - 3b)^2$.
37. $(6x + y)^2$.
38. $(4x - 2y)^2$.
39. $(4x - y^2)^2$.
40. $(2x^3 - y^6)^2$.
41. $(xy + 10)^2$.
42. $(10 + xy)^2$.
43. $(2ab - 5)^2$.
44. $(xy + 2)^2$.
45. $(a - 3mn)^2$.
46. $(B^2 - 4AC)^2$.
47. $(4x + .5)^2$.
48. $(.03x + .04)^2$.
49. 29^2. *Hint:* $29^2 = (30 - 1)^2$.
50. 53^2.
51. 13^2.
52. 104^2.
53. 10.2^2.
54. 39^2.
55. 27^2.
56. 99^2.
57. 37^2.
58. 43^2.
59. 24^2.
60. 1005^2.
61. 99.5^2.
62. 22.5^2.

Optional

63. $(.2 - 4x)^2$.
64. $(.5x - 3y)^2$.
65. $(.3x^2 - .5y)^2$.
66. $(.2a^2 - .5y^2)^2$.
67. $(x^n + 4)^2$.
68. $(x^n - 7)^2$.
69. $(x^n + y^m)^2$.
70. $(e^{2x} + 4)^2$.

71. $(x + \frac{1}{2})^2$.

72. $(x + \frac{1}{4})^2$.

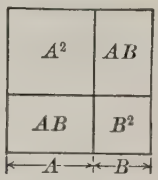
73. $(x - \frac{1}{2})^2$.

74. $\left(\frac{x}{2} + 3\right)^2$.

75. $(\frac{1}{2}\,a + \frac{1}{3}\,b)^2$.

76. $(2\,a - 1)^2 + (2\,a + 1)^2$.

77. Find the value of $(A + B)^2$ in the diagram.



78. Name the binomial whose square is

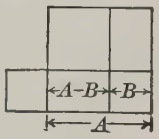
(a) $x^2 + 10\,x + 25$.

(b) $x^2 - 10\,x + 25$.

(c) $x^2 - 16\,x + 64$.

(d) $x^2 + 14\,x + 49$.

79. Show by the diagram that $(A - B)^2 = A^2 - 2\,AB + B^2$.



80. $(a^2 + 2\,ab + b^2) \div (a + b) = ?$

78. Factoring Trinomial Squares. — We should learn to recognize the square of a binomial as soon as we see it. It consists of three terms, two of which are positive and each the square of some quantity. The remaining term may have either sign, but it must be twice the product of those same quantities.

Thus, $(2\,a - b)^2 = 4\,a^2 - 4\,ab + b^2$.

$4\,a^2$ is the square of $2\,a$.

b^2 is the square of $-\,b$.

$-\,4\,ab$ is twice the product of $2\,a$ and $-\,b$.

Oral Exercises

State the binomial of which each of the following trinomials is the square:

1. $a^2 - 6\,a + 9$.
2. $x^2 + 2\,xy + y^2$.
3. $4\,x^2 + 4\,x + 1$.
4. $9\,x^2 - 6\,x + 1$.
5. $25\,x^2 + 10\,xy + y^2$.
6. $4\,x^2 + 12\,xy + 9\,y^2$.
7. $81\,m^2 - 36\,mn + 4\,n^2$.
8. $16\,s^2t^2 + 8\,stu + u^2$.

Factor the following trinomial squares:

9. $l^2 + 4\,lmn + 4\,m^2n^2$.
10. $121\,z^2 - 66\,zs + 9\,s^2$.
11. $a^2 + 2\,ac^2 + c^4$.
12. $x^4 - 6\,x^2y^2 + 9\,y^4$.
13. $y^6 - 6\,y^3 + 9$.
14. $a^4 + 4\,a^2b^2 + 4\,b^4$.
15. $x^4 - 2\,x^2y + y^2$.
16. $9\,x^2 + 30\,x + 25$.
17. $K^2 + 4\,KH + 4\,H^2$.
18. $x^2 + 18\,xy + 81\,y^2$.
19. $1 + 12\,a + 36\,a^2$.
20. $16 - 24\,m^2 + 9\,m^4$.
21. $x^2 - x + \frac{1}{4}$.
22. $16\,a^2 - 48\,a + 36$.
23. $\frac{1}{4}\,h^2 - \frac{1}{3}\,hk + \frac{1}{9}\,k^2$.
24. $.16\,a^2 + .16\,ab + .04\,b^2$.
25. $x^2 + 8\,xy + 16\,y^2$.
26. $a^2 + \frac{2}{3}\,a + \frac{1}{9}$.
27. $9 + 30 + 25$.
28. $4\,x^2 + 1.2\,x + .09$.
29. $x^{2a} + 2\,x^a + 1$.
30. $a^{2m} - 2\,a^mb + b^2$.
31. $x^{2a} + 4\,x^ay^a + 4\,y^{2a}$.
32. $4\,y^{2n} - 12\,y^n + 9$.
33. $(a+b)^2 + 2(a+b) + 1$.
34. $144\,x^2 - 72\,x + 9$.

Insert the proper middle term to make each of the following trinomials a perfect square:

35. $x^2 + \,?\, + y^2$.
36. $4\,k^2 + \,?\, + 1$.
37. $x^2 - \,?\, + 100$.
38. $9\,a^2 - \,?\, + 25\,b^2$.
39. $81 + \,?\, + 25$.
40. $144\,m^2 + \,?\, + 144\,n^2$.
41. $16\,x^2 + \,?\, + 36$.
42. $x^2 - \,?\, + 81$.

Factor the numerator and divide numerator and denominator by the common factor:

43. $\dfrac{4\,x^2 + 20\,xy + 25\,y^2}{2\,x + 5\,y}$.
44. $\dfrac{16\,x^2 + 72\,x + 81}{4\,x + 9}$.
45. $\dfrac{9\,x^2 + 6\,x + 1}{3\,x + 1}$.
46. $\dfrac{1 - 10\,x + 25\,x^2}{1 - 5\,x}$.
47. $\dfrac{x^{2m} + 2\,x^m + 1}{x^m + 1}$.
48. $\dfrac{k^2 - 12\,k + 36}{k - 6}$.

79. Prime Factors. — It often happens that, after having factored the expression, we can factor one or both of the factors again, thus obtaining more than two factors in the final result. We should do this whenever possible, thus obtaining *prime factors*, which were defined in section 70.

In the following exercises you will find some examples of this kind. *You should always examine the expression first to see if there is a common monomial factor,* and if so, remove it, before you attempt to apply any other method.

Review Exercises in Factoring

Factor each of the following, being sure to obtain prime factors:

1. $3x^2 - 27$.
2. $64x^2 - 16y^2$.
3. $ax^3 - a^3x$.
4. $16x^4 - 1$.
5. $x^4 - a^4$.
6. $x^8 - y^4$.
7. $a^8 - a^4$.
8. $m^8 - 1$.
9. $K^2 - K^6$.
10. $m^4n^4 - 81x^4$.
11. $32m^4 - 162a^4b^4$.
12. $4a^2x^2y^3t - 4b^2x^2y^3t$.
13. $3ml^2 - 363m^3$.
14. $243hk - 3h^5k$.
15. $ax^4 - 64a^3$.
16. $m^{12} - n^8$.
17. $x^3 - 16x$.
18. $.4x^5 - .064x^3$.

Optional

19. $x^3 + 8x^2 + 16x$.
20. $x^3 - 14x^2 + 49x$.
21. $9x^2 + 42x + 49$.
22. $6x^2 - 7x - 5$.
23. $3x^2 - 17x - 6$.
24. $3x^2 - 3xy + 9xz$.
25. $ax^2 - 2ax + a$.
26. $2m^2 + 16m + 24$.
27. $2s^2 - 12s + 18$.
28. $6x^3 + 15x^2 - 9x$.
29. $2x^3 - 4x^2 - 6x$.
30. $mx^2 + 2mx - 6m^2x$.
31. $11x^2 - 35x + 6$.
32. $3 - 13x - 10x^2$.

33. $m - 6m^2 + 9m^3$.

34. $m^2 - 3mn + 2n^2$.

35. $6x^2 - 15xy + 6y^2$.

36. $4x^3 - 16x^2 + 12x$.

37. $ax^2 - 2a^2x + a^3$.

38. $ax^3 - ax^2 - 6ax$.

39. $36x^2y + 24xy + 4y$.

40. $4am^2 + 20am + 24a$.

41. $5a^2x - 30ax + 25x$.

42. $mx^3 - 4m^2x^2 + 4m^3x$.

43. $12x^3 - 10x^2 - 42x$.

44. $3x^3 + 30x^2 + 75x$.

45. $5x^3 - 30x^2y + 45xy^2$.

46. $2x^2 - 10x + 12$.

47. $3m^3x^2 + 36m^2x + 108m$.

48. x^2 times a certain quantity is $x^5 + x^4 + x^3 + x^2$. What is the quantity?

49. The area of a rectangle is $4m^3 + 2m^2 - 8m$. Its width is $2m$. What is its length?

50. The area of a rectangle is represented by

$$2m^2 + 9m - 18.$$

If its length is $2m - 3$, what is its perimeter?

51. The area of a square is $x^2 - 6x + 9$. How long is one of the two equal adjacent sides?

True and False Test

Classify each of the twenty statements as true or false.

1. In $3a$, 3 is an exponent.

2. If the length of a rectangle is l and its width is w, its perimeter is $2l + w$.

3. A root of $6x - 3 = 4x + 7$ is 5.

4. $(a + b)^2 = a^2 + ab + b^2$.

5. $a - (b + c) = a - b - c$.

6. $a^2 - b^2 = (a - b)^2$.

7. The factors of $a^2 - 4b^2$ are $(a - 4b)(a + 4b)$.

8. $\$c = \frac{c}{100}$ cents.

9. n feet $= \frac{n}{3}$ yards.

10. If $x = 3$, $2x^2 = 36$.

11. $6 - 2 \cdot 3 = 0$.

12. $15 - 5 \cdot 3 = 30$.

13. $(15 - 5)3 = 30$.

14. $10 - 2 \cdot 3 = 10 - (2 \cdot 3)$.

15. $6x^2 - 10x + 3 = 3x + 10 - 6x^2$ is an identity.

16. $3 - 2x - x^2$ is the same as $x^2 + 2x - 3$.

17. $(a^2 + ab + b^2)(a^2 - ab + b^2) = a^4 + a^2b^2 + b^4$.

18. -5 is a root of $5x = 0$.

19. -2 is a root of $-4x = -8$.

20. $(x + 1) + (x - 1) = 2x$ is a conditional equation.

80. Factoring by Grouping.[1]

Illustrative example I:

Factor $2xy - 3y + 4x - 6$.

Solution: By the use of parentheses we may group the expression so that it will contain only two terms; as,

$$(2xy - 3y) + (4x - 6).$$

From the first parenthesis we may remove the factor y, and from the second the factor 2, obtaining $y(2x - 3) + 2(2x - 3)$. We now see that the common factor $(2x - 3)$ is contained in both terms. This, then, is a factor of the expression. By dividing the expression by the factor already found we obtain the quotient $y + 2$.

Therefore,

$$\begin{aligned} 2xy - 3y + 4x - 6 &= (2xy - 3y) + (4x - 6) \\ &= y(2x - 3) + 2(2x - 3) \\ &= (2x - 3)(y + 2). \end{aligned}$$

Check by multiplying these factors.

[1] The preceding material covers the more important cases of factoring. For schools desiring a more extended treatment of the subject, the cases which follow are optional.

Illustrative example II:

Factor $2x^2 - nx - 2mx + mn$.

Solution: $2x^2 - 2mx - nx + mn$ Arranging.

$= (2x^2 - 2mx) - (nx - mn)$ By grouping.

$= 2x(x - m) - n(x - m)$ Check.

$= (x - m)(2x - n)$.

Exercises

Factor each of the following polynomials:

1. $ac - 2c + ad - 2d$.
2. $xy - ay + bx - ab$.
3. $ax + 2mx - 3ay - 6my$.
4. $5xy + 2my - 15x - 6m$.
5. $2x^2 - 8ax + 3nx - 12an$.
6. $6xy - y + 24x - 4$.
7. $5x^2 - 3x - 10xy + 6y$.
8. $mx + 3nx - 4my - 12ny$.
9. $4x^2 + 2x - 12ax - 6a$.
10. $16am - 12bm + 8an - 6bn$.
11. $15rs^2 + 3r^2s - 45ns^2 - 9rns$.
12. $4kt^2s + tks^2 - 12kt^2w - 3ktws$.
13. $a(a + b) + b(a + b) + 3(a + b)$.
14. $h(a - b) + k(b - a)$.

Hint: This is the same as $h(a - b) - k(a - b)$.

15. $x(y - z) - w(z - y)$.
16. $x^2 + 6x - \frac{5}{2}xy - 15y$.
17. $3a(a + b + c) + 6b(a + b + c) + 9c(a + b + c)$.
18. $2x^2 - 2xy - 3xz + 3yz$.
19. $x^2 + 3x - .5xy - 1.5y$.

81. Grouping to Show Difference of Squares. — In section 74 we discussed factoring the difference of two squares we may use. Similar methods may be used when one of the squares is a trinomial square of the type mentioned in section 78.

Illustrative example I:

Factor $x^2 - 6x + 9 - 4m^2$.

We note that the first three terms represent the square of the binomial $x - 3$. $4m^2$ is also a perfect square. Thus we have the difference of two squares. The work may be indicated as follows:

$$(x^2 - 6x + 9) - 4m^2 = (x - 3)^2 - 4m^2$$
$$= (x - 3 + 2m)(x - 3 - 2m).$$

Check by multiplying the factors.

Illustrative example II:

Factor $x^2 - y^2 + 2yz - z^2$.

Inclosing the last three terms within a parenthesis, preceded by a $-$ sign, being careful to change the signs of the terms inclosed, we obtain:

$$x^2 - (y^2 - 2yz + z^2) = x^2 - (y - z)^2$$
$$= [x + (y - z)][x - (y - z)]$$
$$= (x + y - z)(x - y + z).$$

Check.

Exercises

Factor each of the following expressions:

1. $x^2 - 2xy + y^2 - 16$.
2. $25a^2 + 10ab + b^2 - 9x^2$.
3. $81m^2 - 18mn + n^2 - 36c^2$.
4. $16 - x^2 - 2xy - y^2$.
5. $25a^2 - 4b^2 + 4bc - c^2$.
6. $9m^2 - 16n^2 + 40n - 25$.

7. $121\,a^2 + 22\,a - 49\,b^2 + 1$.
8. $36\,x^2 - 49\,y^2 + 36\,xz + 9\,z^2$.
9. $100 - a^2b^2 - 6\,abc - 9\,c^2$.
10. $s^2 - 10\,s + 25 - 144\,a^2$.
11. $1 - 6\,a - 49\,b^2 + 9\,a^2$.
12. $(x^2 + 2\,x + 1) - (y^2 + 2\,yz - z^2)$.
13. $b^2 - a^2 + 2\,a - 1$.
14. $a^2 - b^2 - c^2 + 2\,bc$.
15. $a^2 - x^2 + 2\,x - 1$.
16. $H^2 + 4\,HK - 16 + 4\,K^2$.
17. $4\,x^2y^4 - 1 + 6\,x - 9\,x^2$.
18. $1 - a^2 + 2\,ab - b^2$.
19. $a^2 - b^2 + 2\,b - 1$.
20. $x^{2a} + 2\,x^a + 1 - 4\,b^{2m}$.
21. $A^2 - 4\,C^2 - 8\,AB + 16\,B^2$.

Exercises

Miscellaneous examples in special products and quotients. Write the product of each of the following:

1. $(2\,a + 3\,m)(a + 3\,m)$.
2. $(3\,x + 2)(5\,x - 7)$.
3. $(2\,a - 3)^2$.
4. $(x - 1)(x + 16)$.
5. $(x + 6)(x - 6)$.
6. $(x + 2\,y)(3\,x - 5\,y)$.
7. $(7\,y + 3\,c)(2\,y - c)$.
8. 81^2.
9. $(3\,a - 2)^2$.
10. $69 \cdot 71$.
11. $ab(ax + 2\,x + 3)$.
12. $(m + n^2)(m - n^2)$.
13. $6\,a(a - 6)$.
14. $2(a + b)^2$.
15. $[(a + b) - c][(a + b) + c]$.
16. $(3\,a - 7)(2\,a + 5)$.
17. 48^2.
18. $101 \cdot 99$.
19. $(3\,m + n)^2$.

20. $(3a^2-6m)(5a^2+2m)$.

21. $\dfrac{x^2-7x+10}{x-2} = ?$

22. $(x+y^2)^2$.

23. $(3x^2-y^3)^2$.

24. $\dfrac{6x^2+7x-3}{2x+3} = ?$

25. $\dfrac{a^2-9b^2}{a+3b} = ?$

26. $\dfrac{x^2-(y^2+1)^2}{x+y^2+1} = ?$

27. $\dfrac{(a-b)^2-x^2}{a-b-x} = ?$

28. $\dfrac{x^2-10xy+25y^2}{x-5y} = ?$

29. $\dfrac{4x^4-y^4}{2x^2+y^2} = ?$

30. $\dfrac{8x^2+26x-7}{4x-1} = ?$

31. $[(m+n)(m-n)]^2 = ?$

82. Summary of Factoring. — The following hints will help in almost all cases of factoring.

1. Remove common monomial factors.
2. Given a binomial, see if it is the difference of squares.
3. Given a trinomial, see if it can be factored.
4. Be certain that all factors are prime.
5. Check by multiplying the factors.

Miscellaneous Exercises in Factoring

1. ax^2-2a^2x-3a.
2. $4b^3+6ab-10b^2$.
3. $ax^2-2a^2x+a^3$.
4. $3x^2-27$.
5. $2a^3-32a$.
6. $a^2-3a-10$.
7. $4n^2-20nm+25m^2$.
8. $3a^2-3a-6$.
9. $2x^3-8x^2+8x$.
10. x^4-16x^2.
11. x^4-a^2.
12. $16a^4-4$.
13. ax^3-ax^2-6ax.
14. by^3-by^2-2by.
15. a^4-x^4.
16. $2x^2-10x+12$.
17. $64x^2-16y^2$.
18. $mx^3-4m^2x^2+4m^3x$.
19. $5a^2x-30ax+25x$.
20. $4am^2+20am+24a$.

21. $9x^2 - 16y^2$.

22. $c^2 + 3c - 54$.

23. $r^2s^2 - 4r^4s^4$.

24. $a^2 - a - 12$.

25. $8 - 18m^2$.

26. $6a^2 + 7a + 2$.

27. $x^4 - x^2$.

28. $m^2 - 11m + 18$.

29. $6p^2 - pq - q^2$.

30. $2a^2 - ab - b^2$.

31. $2a^2 + 5ab - 3b^2$.

32. $x^2 - 25y^2z^2$.

33. $m^2 + m - 12$.

34. $9a^3 - 9a$.

35. $4s^2 - 20sw + 25w^2$.

36. $25a^2b^2 - 16a^4b^2$.

37. $1 + 6ab + 9a^2b^2$.

38. $a^2 - 10a + 16$.

39. $x^2 - 7x - 18$.

40. $81a^2 + 18ab + b^2$.

41. $2c^3 - 20c^2 + 50c$.

42. $a^4 - 16$.

43. $80r - 40r^2 + 5r^3$.

44. $x^2 - x - 72$.

45. $5x^3 - 20xy^2$.

46. $5hk + k^2 - 36h^2$.

47. $121x^2 - 49y^2$.

48. $2x^2 + 10x - 48$.

49. $a^4 - 14a^2 + 49$.

50. $81 - c^4$.

51. $5c^3 + 15c^2 - 140c$.

52. $2x^2 + 13x + 18$.

53. $x^5 - xy^4$.

54. $4l^2 + 20l + 21$.

55. $2 + x - 15x^2$.

56. $4x^2 + 10x + 4$.

57. $x^3 - 9x$.

58. $x^2 + x^3 - 6x^4$.

59. $5a^3 - 10a^2b - 75ab^2$.

60. $14m^2 - 39m + 10$.

61. $3x^2 - 10xy + 3y^2$.

62. $6a^2x + 7ax - 5x$.

63. $16 - 9a^2b^2x^4$.

64. $98c + 28c^2 + 2c^3$.

65. $4(x - 9)^2 - 9a^2$.

66. $9ac^2 - 64ad^2$.

67. $36a^2 + 23a - 3$.

68. $x^6 - 15x^3 + 36$.

69. $7x^3 - 35x^2 - 252x$.

70. $a^2 - 2ab - c^2 + b^2$.

71. $a^2 + 2ab + b^2 - c^2$.

72. $x^2 - 6r - 9r^2 - 1$.

73. $x^4 + 2x^2y^2 + y^4 - a^2b^2$.

74. $4a^2 - 12ab + 9b^2 - 9c^4$.

75. $(a - c)^2 - 4(a - c)x + 4x^2$.

76. $\frac{1}{3}a(x + y) + \frac{1}{3}(x + y)$.

77. $e^{2m} - h^{4n}$.

78. $.4\,x^4 - .4\,y^4$.

79. $9\,e^z h^2 - e^z$.

80. $.7\,x - 2.1\,y + 4.2\,z$.

Now that we have mastered the subject of factoring we are ready to apply it to the solution of equations.

83. Factoring in Literal Equations. — One method by which we solve literal equations may be shown by the following

Illustrative example:

Solve $ax + b^2 = a^2 - bx$.

Solution:

$ax + bx = a^2 - b^2$.	By transposing certain terms.
$x(a + b) = (a + b)(a - b)$.	Factoring both members.
$x = a - b$, root.	Dividing both members by $(a + b)$.

Check:

$a(a - b) + b^2 = a^2 - b(a - b)$.	Substituting $a - b$ for x in the original equation.
$a^2 - ab + b^2 = a^2 - ab + b^2$.	

Exercises

Solve each of the following equations for x or y and check as directed by the teacher:

1. $mx - nx = m^2 - n^2$.
2. $nx - x = n^2 - 1$.
3. $klx + 3\,x = 6\,kl + 18$.
4. $a(x - a) = b(x - b)$.
5. $xm + an^2 = xn + am^2$.
6. $ax + a^2 = -x - a$.
7. $2\,bx - sb = 3\,s - 6\,x$.

8. $bx - b^2 = cx - c^2$.
9. $ax + bx = a^2 + 2\,ab + b^2$.
10. $bx + cx = b^2 + 3\,bc + 2\,c^2$.
11. $ax - 2\,x = a^2 + a - 6$.
12. $mx + 2\,nx + x = am + 2\,an + a$.
13. $2 + 2\,cx = 2\,c^2 + x + 3\,c$.
14. $10\,m - 25 - 5\,x = m^2 - mx$.
15. $bx - 4\,ax = b^2 - 8\,ab + 16\,a^2$.
16. $x - 3\,bx = 1 - 6\,b + 9\,b^2$.
17. $3\,bx + 4\,ax = 9\,b^2 + 24\,ba + 16\,a^2$.
18. $ax - 2\,a + 3 = a^2 + x$.
19. $b + 42 - b^2 = 7\,x - bx$.
20. $cx - c^2 = 3\,x - 5\,c + 6$.
21. $a^2 + a = ay - 5\,y + 30$.
22. $2\,x = 10 - 3\,a + ax - a^2$.
23. $ny = n^2 + y - 7 + 6\,n$.
24. $ax - a^2 + 10\,b^2 = -3\,ab - 2\,bx$.
25. $8\,n^2 - 5\,kn - 3\,k^2 + kx = nx$.

84. Solving Quadratic Equations by Factoring.— A *quadratic* equation is an equation which contains the second power of the unknown quantity, but no higher power. For example:

$x^2 - bx = c^2$; $x^2 = 9$; $2\,x^2 + 6\,x = 10$; $x^2 - 4\,x + 3 = 0$; are quadratic equations.

Quadratic equations as well as literal equations may often be solved by factoring. In many cases we require more advanced methods, which are explained in a later chapter. But the method of solving by factoring where it is possible is shown by the following

Illustrative example:

Solve $x^2 - 3x = 28$.

Solution: $x^2 - 3x - 28 = 0$. Transposing the 28.

$(x - 7)(x + 4) = 0$. Factoring the left member.

$x - 7 = 0$	$x + 4 = 0$.
$x = 7$	$x = -4$.

By setting each factor equal to 0, we obtain two simple equations and solve each.

Both of these answers, 7 and -4, will check in the equation, for $49 - 21 = 28$ and $16 - (-12) = 28$, therefore both are roots. It is important for you to remember that *a quadratic equation always has two roots.*

Explanation. The expression $x^2 - 3x - 28$ is a product, since it is made up of the factors $(x - 7)$ and $(x + 4)$. The equation states that this product equals 0. No product can equal 0 unless one of the factors equals 0, but the product will equal 0 if any factor in it equals 0.

Therefore, in this product either $x - 7 = 0$ or $x + 4 = 0$, and both of these conditions will satisfy the equation.

If $x - 7 = 0$, $x = 7$; if $x + 4 = 0$, $x = -4$; and these are the roots of the equation.

The above solution illustrates the following

Rule. — *To solve a quadratic equation by factoring,*

(1) *Transpose every term to the left member of the equation, thus making the right member 0.*

(2) *Factor the left member.*

(3) *Set each factor which contains the unknown quantity equal to 0, and solve the resulting simple equations.*

The results obtained are the roots of the quadratic equation.

Exercises

By the preceding rule solve each of the following equations. Check as your teacher directs.

1. $x^2 + 7x + 12 = 0$.
2. $x^2 + 9x + 20 = 0$.
3. $x^2 + 6x + 5 = 0$.
4. $x^2 - 5x + 6 = 0$.
5. $x^2 - x - 20 = 0$.
6. $x^2 - 11x + 30 = 0$.
7. $x^2 + 3x - 40 = 0$.
8. $x^2 - 13x + 36 = 0$.
9. $x^2 - 15x + 50 = 0$.
10. $x^2 + 6x - 16 = 0$.
11. $x^2 + 2ax - 3a^2 = 0$.
12. $x^2 - 4a^2b^2 = 0$.
13. $y^2 - a^2 = 0$.
14. $x^2 - 6x = 0$.
15. $3x^2 - 21x = 0$.
16. $6x^2 + 10x - 4 = 0$.
17. $6x^2 + 25x + 14 = 0$.
18. $3x^2 + 4bx + b^2 = 0$.
19. $2x^2 + 5x + 2 = 0$.
20. $x^2 + 5x = -4$.
21. $x^2 + 9x = -18$.
22. $x^2 + 12x = -27$.
23. $x^2 + 2x = 3$.
24. $5x^2 = 40x$.
25. $2x^2 = 3 + 5x$.
26. $2x^2 = 5 - 3x$.
27. $6x^2 = 12 - x$.
28. $4x^2 - 29x = -7$.
29. $2x^2 + x = 21$.
30. $y = 15 - 2y^2$.
31. $8y^2 + 2y = 3$.
32. $12x^2 + 13x = 14$.
33. $12y^2 = 2 - 10y$.
34. $2x^2 = 10a^2 + ax$.
35. $y^2 + 4by = 12b^2$.
36. $a^2 - 4ax = 5x^2$.

37. Find three consecutive numbers, of which x is the middle one, such that if the product of the first by the last is increased by the middle number, the result is 29.

38. Find two consecutive whole numbers whose product equals 6.

39. The square of a certain number plus twice the number equals 35. Find the number.

40. The square of a certain number diminished by five times the number equals 24. Find the number.

41. Three times the square of a certain number diminished by four times the number equals 4. Find the number.

42. A certain number multiplied by 5 times the number equals 125. Find the number.

43. If 40 is added to the square of a number, the sum is equal to thirteen times the number. What is the number?

44. The length of a certain rectangle is 3 times its width. The area of the rectangle is 48. Find the width and the length.

Form the equation whose roots are:

45. 2 and 3.

Solution: Corresponding to these roots we have the factors $x - 2$ and $x - 3$, hence the equation is

$$(x - 2)(x - 3) = 0,$$

$$\text{or, } x^2 - 5x + 6 = 0.$$

46. 5 and 3.

47. 0 and 3.

48. 2, -1, and -2.

49. 4, 2, and 0.

50. a, b, and c.

Oral Algebraic Expression Review Exercises

1. How many hours will it take a man to travel n miles at the rate of m miles per hour?

2. What is the perimeter of a square whose side is 6? Whose side is x? Whose side is $x + 1$?

3. By how much does $h + k$ exceed $2k$? Exceed $3h$? Exceed 7?

4. Draw a rectangle and mark the length $x + 3$ and the width $x + 1$. What is the area? What is the area if $x = 6$?

5. What is the result if the square of x plus the square of y is subtracted from the square of $(x + y)$?

6. What is the value of $(x + y)^2$? $(20 + 6)^2$? $(15 - 2)^2$?

7. Write the following as an equation: three times y exceeds 18 by 5.

8. What is the next even number below $4\,x$? Above $4\,x$?

9. State the interest (i) on d dollars at p per cent for y years as an equation.

10. A stick is 10 feet long. 3 feet is cut off. State the part that is left by the use of the minus sign.

A Fishing Schooner

11. If a line is 10 inches long and one part of the line is represented by x, what represents the other part?

12. Five times the square of a certain number plus four times the number equals 26. State this by the use of algebraic symbols in an equation.

13. $6\,x^2 - 5\,x = 28$. State this by the use of words.

14. What is the sum of four consecutive even numbers the greatest of which is $x + 10$?

15. How long will it take a fishing schooner to sail m miles if it travels 10 miles per hour? If it sails x miles per hour?

16. A room is y yards long and f feet wide. What is its area in square feet? In square yards?

17. What is the sum of $4\,x$ multiplied by $(x + 6)$ and $5\,x$ multiplied by $(x + 4)$?

18. In y years a man will be 27 years of age. What is his present age?

19. What is the length of one side of an equilateral triangle (a triangle with all sides equal) if its perimeter is denoted by $6x + 9$? By $.12y - .15$? By $24H + 36K$?

20. What is the length of an edge of a cube whose volume is 8? x^3? $(x + 3)^3$? 2^3? 6^3?

21. What is the length of an edge of a cube if its complete outside surface is 24? 96? $6x^2$? $6(x + 1)^2$?

22. What must be subtracted from $h + 2k$ to leave $2h - 3k$?

23. From the sum of $3x$ and $4y$ subtract their difference.

24. What is the circumference of a circle whose diameter is 4 feet, if $C_0 = 2\pi r$, and $\pi = \frac{22}{7}$? If the diameter is 14 inches? If the diameter is $x + 28$ feet?

25. Without solving, tell whether the value of x is positive or negative and why:

(a) $4x + 6x + 2x + 3x = 36$.

(b) $+2x + 4x + 6x + 7x = -114$.

Written Review Exercises

1. Multiply $2 + a - a^2$ by $a^2 - a - 2$ and check your result.

2. Factor $a^{2n} + 6a^n + 9$.

3. Find two numbers whose sum is 58 and whose difference is 28.

4. Simplify: $x^2 - \{x + x^2 + (x - 2x^2)\} - x$.

5. The first angle of a triangle is 40° less than the second angle. The third angle is 40° more than the sum of the first and second angles. The sum of all the angles is 180°. Find the value of each angle.

6. Using the difference of two numbers times their sum, find the cost of 116 articles at \$124 each.

7. Express by the use of algebraic symbols: a times b times c and the result divided by the sum of x and y.

What is the value of your expression if $a = 1, b = 2, c = 3, x = 4$, and $y = 8$?

8. Factor each of the following:

(a) $a^4 - 2a^2 + 1$. (b) $12m^2 - 7m + 1$. (c) $a^{2m} - 1$.

9. If a cubic foot of iron weighs three times as much as one cubic foot of aluminum, and six cubic feet of aluminum added to four cubic feet of iron weigh 2880 pounds, what is the weight of one cubic foot of each?

10. By how much does $3x + 25$ exceed $2x - 15$?

11. 8% of what number equals 64?

12. What is the value of $a^2 - 3(a - b)4(a - 2b)$ if $a = 4$ and $b = 5$?

Write the following as equations:

13. b subtracted from a and the result increased by 9 equals h.

14. Ten is as much below x as 15 is above x.

15. Three times x increased by 9 equals five times x diminished by 6.

16. Five times the difference of x and y exceeds z by as much as w exceeds 8.

17. Does $x = 3$ satisfy the equation $x^2 + 4x = 21$? $x^2 - 5x = 16$?

18. Three times a certain number increased by 17.16 equals five times the number diminished by 8.04. Find the number.

19. Which is greater and by how much, $a^3 + b^3$ or $(a + b)^3$ if $a = 3$ and $b = 4$?

20. How many cents will we have if we subtract d dimes from the sum of q quarters and twenty cents?

21. What is the price of one orange if 6 sell for a quarter? If 6 sell for c cents? If 12 sell for d dimes plus c cents?

22. $\dfrac{.04\,x^2 - .25\,y^2}{.2\,x + .5\,y} = ?$

23. Factor:

(a) $6\,x^2 + 13\,xy + 6\,y^2$.
(b) $a^2 - ab - 2\,b^2$.
(c) $6\,x^2 - 6\,y^2$.
(d) $2\,x^2 - 32$.
(e) $8\,x^2 + 34\,x + 15$.
(f) $y^2 - 6\,y - 247$.

24. Air is composed of a mixture of one part of oxygen and four parts of nitrogen. How many cubic feet of oxygen in a room six yards long, four yards wide, and twelve feet high?

25. State a rule to show whether a binomial is or is not the product of the sum and difference of the same two numbers.

26. Find the area of a rectangle whose length is $5\,x + 7$ and whose width is $4\,x - 2$. What is the area if $x = 6$?

27. Find the volume of a rectangular solid whose edges are respectively x, $x + 1$, and $x + 2$.

What is the total outside surface of this solid?

What is the total length of all edges?

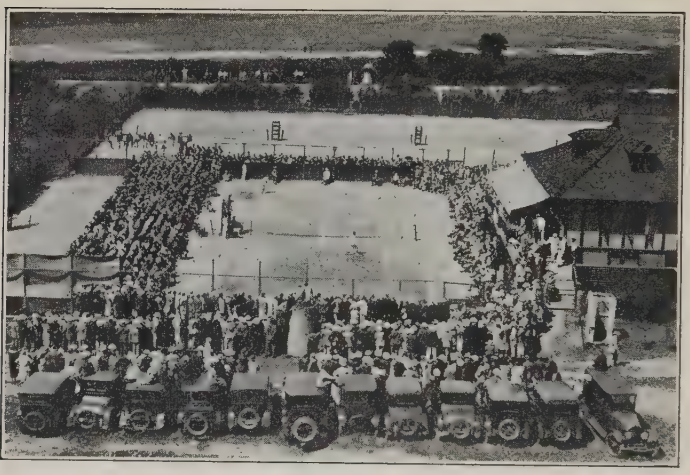
Find each of the above values when $x = 3$.

28. Divide 1 by $1 + x$ to five terms in your quotient.

29. In a certain high school the number of pupils in the four classes was distributed as follows:

First year	690 pupils	Third year	420 pupils
Second year	750 pupils	Fourth year	300 pupils

Represent this information by the use of the bar or circle graph.



A Single Tennis Court

30. A single tennis court is 51 feet longer than its width. A double tennis court is 9 feet wider than a single court and the same length. The area of the double court is 702 square feet more than that of the single court. Find the dimensions of each court.

CHAPTER VIII

ALGEBRAIC FRACTIONS

Up to this time we have avoided fractions in our discussion of algebraic problems, except when we used them to indicate division and in a few simple equations having terms with fractional coefficients. However, fractions occur frequently in algebra and a mastery of them is necessary in our further study of equations. In the present chapter we shall see how our knowledge of factoring helps us to simplify fractional expressions.

85. An Algebraic Fraction is the indicated quotient of two algebraic expressions, as $\frac{x}{y}, \frac{2\,ab}{3\,x}$. It is said to be in its *lowest terms* when there are no factors common to both numerator and denominator.

86. Reduction to Lowest Terms; Highest Common Factor. — In arithmetic when we are asked to reduce to lowest terms a fraction like $\frac{32}{48}$, we do so by dividing both numerator and denominator by their greatest common divisor, or, as we call it in algebra, their *highest common factor*. This means the *largest number contained in both of them.*

Evidently the largest number exactly contained in both 32 and 48 is 16. As we know from arithmetic *it does not change the value of a fraction to divide both numerator and denominator by the same number.* If, then, we divide both numerator and denominator of the given fraction by 16, we obtain $\frac{2}{3}$, which is the form of the fraction when reduced to lowest terms.

There is a second method by which we might reduce this fraction to its lowest terms:

$$\frac{32}{48} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = \frac{2}{3}.$$

By this method both numerator and denominator have first been factored into their prime factors, and then each divided by every common factor. The product of all prime factors common to both is the highest common factor. Thus we have really divided both numerator and denominator by the highest common factor, even though we have done it in separate steps.

In reducing algebraic fractions, the first of these methods is to be preferred when the numerator and denominator are monomials, and the second is to be preferred when they contain two or more terms.

Illustrative example I:

Reduce to lowest terms $\frac{12\,a^3x^5y}{15\,a^2x^2z}$.

Solution: The H. C. F. (highest common factor) of numerator and denominator is $3\,a^2x^2$. Dividing both numerator and denominator by $3\,a^2x^2$, we obtain $\frac{4\,ax^3y}{5\,z}$, which is the form of the fraction when in lowest terms.

Illustrative example II:

Reduce to lowest terms $\frac{x^2 - 9}{x^2 - x - 6}$.

Solution: Factoring numerator and denominator into prime factors, and dividing both numerator and denominator by the common factors, we obtain

$$\frac{x^2 - 9}{x^2 - x - 6} = \frac{(x + 3)\overset{1}{\cancel{(x - 3)}}}{(x + 2)\underset{1}{\cancel{(x - 3)}}} = \frac{x + 3}{x + 2}.$$

Here we have divided both numerator and denominator by $x - 3$.

Check by letting $x = 4$.

$$\frac{x^2 - 9}{x^2 - x - 6} = \frac{16 - 9}{16 - 4 - 6} = \frac{7}{6}.$$

$$\frac{x + 3}{x + 2} = \frac{4 + 3}{4 + 2} = \frac{7}{6}.$$

We reduce fractions to their lowest terms by use of the following

Rule. — (1) *If the numerator and denominator are monomials, divide each by their highest common factor.*

(2) *If they are polynomials, factor each into its prime factors and divide numerator and denominator by all factors common to both.*

In applying the above rule you must be careful to divide out *factors* and not *terms*. What is the distinction between factors and terms?

Exercises

Reduce each of the following fractions to lowest terms:

1. $\frac{8x}{16x^2}$.

2. $\frac{9a^2x}{27ax}$.

3. $\frac{4bc}{12abc}$.

4. $\frac{15c^2x}{5cx^2}$.

5. $\frac{6ab^3}{9a^2}$.

6. $\frac{7m^2n^2}{14m^3n}$.

7. $\frac{11a^3b^3}{22ab^2}$.

8. $\frac{25a^4b^2}{15a^3b}$.

9. $\frac{18abc^2}{9a^2bc}$.

10. $\frac{-x^3}{x^2}$.

11. $\frac{-4x^2}{8x}$.

12. $\frac{16ab^3}{-8a^2b^2}$.

13. $\frac{12x^3y}{-4xy^2}$.

14. $\frac{-18ac}{-27ab}$.

15. $\frac{-20m^2n^2p}{4mnp}$.

16. $\dfrac{18\,a^3bc^4}{24\,a^2b^3}$.

17. $\dfrac{20\,m^2nx}{45\,mnx^2}$.

18. $\dfrac{36\,abc^3x^2}{84\,bc^3x}$.

19. $\dfrac{33\,d^2ef}{121\,dek}$.

20. $\dfrac{96\,x^2yz^4}{144\,y^3z^5}$.

21. $\dfrac{4\,\pi R^2}{\pi R^3}$.

22. $\dfrac{\frac{4}{3}\pi r^3}{\frac{1}{3}\pi r^2}$.

23. $\dfrac{3\,x^2y^2}{3\,x(x-y)}$.

24. $\dfrac{5\,m^2(m-n)}{25\,mn}$.

25. $\dfrac{15\,a^2b^2(c-d)}{45\,a^3b(c-d)}$.

26. $\dfrac{2(c-d)^2}{8(c-d)}$.

27. $\dfrac{9(m-n)}{6(m-n)^2}$.

28. $\dfrac{16\,a^2(x-y)^2}{8\,a^3(x-y)^3}$.

29. $\dfrac{18\,x^3(x-y)}{45\,x^2y^2(x+y)}$.

30. $\dfrac{4\,xy^2(x-y)^2}{24\,x^3y^3(x-y)^3}$.

31. $\dfrac{4\,xy^3z}{4\,xz+4\,yz}$.

32. $\dfrac{(m-n)^2}{m^2-n^2}$.

33. $\dfrac{m^2-9\,n^2}{2\,am+6\,an}$.

34. $\dfrac{y^2-1}{4\,y+4}$.

35. $\dfrac{6\,a^2+2\,ab}{9\,a^2-b^2}$.

36. $\dfrac{15\,y-9}{6-10\,y}$.

37. $\dfrac{6\,ab(4\,c^2-d^2)}{6\,bc+3\,bd}$.

38. $\dfrac{16\,x^2y(m+n)^2}{xm^2-xn^2}$.

39. $\dfrac{b^3-bc^2}{bc(b-c)^2}$.

40. $\dfrac{5-80\,c^2}{10\,a+40\,ac}$.

Optional

41. $\dfrac{x^2-4}{x^2+x-6}$.

42. $\dfrac{y^2-1}{y^2+3\,y+2}$.

43. $\dfrac{x^2-3\,x-4}{x^2+5\,x+4}$.

44. $\dfrac{x^2-6\,x+9}{x^2+2\,x-15}$.

45. $\dfrac{x^2+2\,x-15}{x^2-x-6}$.

46. $\dfrac{x^2-8\,x-9}{x^2-9\,x-10}$.

47. $\dfrac{x^2 - 13x + 22}{x^2 - 9x - 22}$.

48. $\dfrac{16x^2 - 8x + 1}{4x^2 + 11x - 3}$.

49. $\dfrac{2x^2 + x - 3}{6x^2 + 5x - 6}$.

50. $\dfrac{x^2 + 4x - 21}{2x^2 - 5x - 3}$.

51. $\dfrac{a^2 + 6ab + 9b^2}{a^2 + 5ab + 6b^2}$.

State the correct quantity which should be substituted for the question mark in each of the following equations to make them true equations:

52. $\dfrac{5}{6} = \dfrac{10}{?}$.

53. $\dfrac{?}{3} = \dfrac{4}{6}$.

54. $\dfrac{3+4}{8+4} = \dfrac{2+?}{?+9}$.

55. $\dfrac{8+2}{8+5} = ?$ Is the result the same if we cross cut the 8 in both the numerator and the denominator?

56. $\dfrac{8 \cdot 2}{8 \cdot 5} = ?$ Is the result the same if we divide both the numerator and the denominator by 8?

57. When may we cross out a quantity in the numerator and the denominator and not change the value of the fraction?

58. Reduce the fraction $\dfrac{(h+k)^2}{h^2 - k^2}$ to its lowest terms.

Substitute 8 for h and 4 for k in your answer and also in the original fraction. How do the results compare?

87. Multiplication of Fractions. — As in arithmetic fractions are multiplied by first multiplying the numerators to obtain the numerator of the answer, and then multiplying the denominators to obtain the denominator of the answer. If there are common factors in numerator and denominator, these should be divided out before the numerators or de-

nominators are multiplied. Otherwise the result will not be in its lowest terms.

Thus: $$\frac{cx}{3} \cdot \frac{5}{ay} = \frac{5\,cx}{3\,ay}.$$

Again, $$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \cdot \frac{\overset{1}{\cancel{3}}}{\underset{4}{\cancel{8}}} = \frac{1}{4},$$

and $$\frac{\overset{x}{\cancel{ax^2}}}{\underset{y^2}{\cancel{y^3}}} \cdot \frac{\overset{y}{\cancel{y^3}}}{\underset{x}{\cancel{ax}}} \cdot \frac{2}{\underset{x}{\cancel{x}}} = 2\,y.$$

If any numerator or denominator is a polynomial, it must be factored before any division can be performed, as only factors may be divided out of numerator and denominator.

Illustrative example

$$\frac{a-b}{a^2-b^2} \cdot \frac{a^2+ab}{a^2-2\,ab}$$

$$= \frac{\cancel{a-b}}{\cancel{(a+b)}\cancel{(a-b)}} \cdot \frac{\cancel{a}(a+b)}{\cancel{a}(a-2\,b)}$$

$$= \frac{1}{(a-2\,b)}.$$

Check with $a = 3$ and $b = 2$ in original and in answer.

The foregoing illustrations show that fractions may be multiplied by the following

Rule.—(1) *Reduce each integral expression to fractional form.*

(2) *Factor every polynomial numerator or denominator which is not already prime.*

(3) *Divide numerator and denominator by the common factors used the same number of times in each.*

(4) *Multiply the remaining factors in the numerators for the numerator of the product and the remaining factors in the denominators for the denominator of the product.*

NOTE. This multiplication may be merely indicated.

Oral Exercise

Multiply:

1. $\frac{1}{2} \cdot \frac{2}{3}$.
2. $\frac{3}{8} \cdot \frac{16}{9}$.
3. $\frac{5}{7} \cdot \frac{14}{15}$.
4. $\frac{11}{2} \cdot \frac{10}{33}$.
5. $\frac{3}{x} \cdot \frac{x^2}{6}$.
6. $\frac{5}{m^2} \cdot \frac{m}{10}$.
7. $\frac{a}{x} \cdot \frac{3x}{5a^2}$.
8. $\frac{2a}{b} \cdot \frac{3b^2}{4}$.
9. $\frac{x^2}{y} \cdot \frac{y^2}{2x}$.
10. $\frac{4x}{3} \cdot \frac{9}{16x^2}$.
11. $\frac{x}{3m} \cdot \frac{9m^2}{x}$.
12. $\frac{ab^2}{c} \cdot \frac{5c}{3ab}$.
13. $-\frac{3}{4} \cdot 8$.
14. $\frac{5}{6} \cdot 5$.
15. $\frac{.4}{.7} \cdot .14$.
16. $\frac{H+K}{6} \cdot 18$.
17. $\frac{M+N}{M-N} \cdot \frac{M^2-N^2}{(M+N)^2}$.
18. $\frac{1}{(a-b)^2} \cdot \frac{(a-b)^3}{a+b}$.

Exercises

Multiply:

1. $\frac{18}{55} \cdot \frac{11}{9} \cdot \frac{5}{2}$.
2. $\frac{32}{121} \cdot \frac{33}{144}$.
3. $\frac{48}{9} \cdot \frac{54}{120} \cdot \frac{8}{3}$.
4. $\frac{125}{4x^3} \cdot \frac{8x}{25} \cdot \frac{2x^3}{15}$.
5. $\frac{8m^2}{x^2} \cdot \frac{5mx}{4} \cdot \frac{2x}{m^3}$.
6. $\frac{144x^2}{35} \cdot \frac{49}{xy^2} \cdot \frac{y^2}{63}$.
7. $\left(\frac{2x}{3}\right)^2 \cdot \frac{3y}{4x^2} \cdot \frac{6}{y}$.
8. $\left(\frac{5x}{9}\right)^2 \cdot \left(\frac{3}{x}\right)^3 \cdot \frac{2}{3}$.
9. $10 \cdot \frac{3}{x^2} \cdot \frac{x}{5y} \cdot \frac{2y}{3}$.
10. $5\frac{2}{3} \cdot \frac{2}{17} \cdot 1\frac{1}{3}$.
11. $\frac{16x^2y^3}{5xz^4} \cdot \frac{25z^2}{64xy^4} \cdot \frac{10xy^3}{3z}$.
12. $\frac{16a^2b^3}{9c^3} \cdot \frac{27c^2}{8ab}$.

13. $\frac{2\,mxy^2}{5\,ab^3} \cdot \frac{25\,ax^2}{4\,my} \cdot \frac{6\,b^3}{5}$.

14. $\frac{14\,x^2}{5\,y} \cdot \frac{10\,x}{2\,y^2} \cdot \frac{y^3}{x^3}$.

15. $\frac{72\,a^4b}{11\,cd^3} \cdot \frac{44\,c^2d}{108\,a^3}$.

16. $\frac{110\,x^2}{3\,y^3} \cdot \frac{6\,y^2}{5\,x^4} \cdot \frac{x^2y}{11}$.

17. $\frac{35\,m^2n}{6\,a} \cdot \frac{12\,a^2}{7\,mn^2} \cdot \frac{3\,n^3}{5}$.

18. $\frac{64}{125\,x^3} \cdot \frac{50\,ax^2}{16\,a} \cdot \frac{10}{3}$.

19. $\frac{7\,b^2}{8\,c^3} \cdot \frac{56}{2\,b^2c} \cdot \left(\frac{7\,c^2}{3}\right)^2$.

20. $-\frac{5\,x^2}{6} \cdot \frac{12\,a^3}{25\,x} \cdot \frac{110\,a}{9\,x^2}$.

21. $\left(-\frac{225\,a^3x^3}{10\,m^2}\right)\left(-\frac{16\,mn}{45\,ax^4}\right)$.

22. $\frac{81\,m^3ny}{16\,a^4} \cdot \frac{8\,a^2y}{27\,m^2n} \cdot \left(-\frac{2\,n}{y^2}\right)$.

23. $6\frac{3}{4} \cdot \frac{8}{9} \cdot \frac{12\,x}{5\,y} \cdot \frac{5\,y^2}{18}$.

24. $\frac{2\,xy}{3} \cdot \frac{12\,xy}{25} \cdot \frac{5\,a}{6\,x^2y^2} \cdot \frac{1}{2}$.

25. $\left(\frac{-3\,x^2y}{16\,m^3}\right)\left(\frac{-44\,a^3m}{9\,xn}\right)\left(\frac{-8\,a}{11\,n^2}\right)$.

26. $\frac{125\,\pi}{16} \cdot \frac{16\,a}{\pi^2} \cdot \frac{3\,\pi a^2}{5}$.

27. $\frac{8(x-y)}{(x+y)} \cdot \frac{3(x+y)}{16}$.

28. $\frac{5(x-3)}{2(x+1)} \cdot \frac{4(x+1)(x+2)}{15(x-3)}$.

29. $\frac{2\,x^2+6}{5\,y^3} \cdot \frac{10\,y^4}{3\,x^2+9}$.

30. $\frac{x^2-16}{x+3} \cdot \frac{x^2-9}{4\,x-16}$.

31. $\frac{(x-2)^3}{x(a+1)^2} \cdot \frac{a+1}{(x-2)^2}$.

32. $\frac{y^4-16}{(y^2-4)^2} \cdot \frac{y^2-4}{y^2+4}$.

33. $\frac{x^2-81}{3\,x+9} \cdot \frac{x+3}{x-9}$.

34. $\frac{x^2-y^2}{3\,x^2y} \cdot \frac{x^2(x-y)}{x+y}$.

88. Division of Fractions. Reciprocals. — Division of fractions is performed, as in arithmetic, by the following

Rule. — *After having reduced each of the expressions to a simple fraction, invert the divisor and proceed according to the rule for multiplication of fractions.*

Illustrative example I:

$$\frac{16\,a^2x}{5\,m^3} \div \frac{8\,ax^2}{10\,m^2} = \frac{\overset{2\,a}{\cancel{16\,a^2x}}}{\underset{m}{\cancel{5\,m^3}}} \cdot \frac{\overset{2}{\cancel{10\,m^2}}}{\underset{x}{\cancel{8\,ax^2}}} = \frac{4\,a}{mx}.$$

Illustrative example II:

$$\frac{x^2 - 25}{x^2 + 5\,x} \div \frac{x + 5}{x} = \frac{\cancel{(x+5)}(x-5)}{x\cancel{(x+5)}} \cdot \frac{\cancel{x}}{\cancel{x+5}}$$

$$= \frac{x-5}{x+5}. \quad \textit{Answer.}$$

More than one divisor may occur in the same example. The fraction immediately following a division sign must be inverted. Thus in the example

$$\frac{3}{8} \div \frac{5}{4} \div \frac{15}{2}, \text{ both } \frac{5}{4} \text{ and } \frac{15}{2}$$

must be inverted, while in the example

$$\frac{3}{x} \div \frac{4}{x^2} \cdot \frac{8}{9}, \text{ only } \frac{4}{x^2}$$

is inverted.

How would the meaning of the latter example be changed if it were written $\frac{3}{x} \div \left(\frac{4}{x^2} \cdot \frac{8}{9}\right)$? Test both expressions when $x = 3$.

The *reciprocal* of a number is 1 divided by the number. For example: the reciprocal of 3 is $\frac{1}{3}$; of x is $\frac{1}{x}$; of $\frac{1}{x}$ is $1 \div \frac{1}{x} = 1 \cdot \frac{x}{1} = x$.

Find the reciprocal of

$$4,\ \frac{1}{5},\ \frac{2}{x},\ \frac{3}{2x},\ \frac{a}{b},\ \frac{y}{x},\ \frac{2a}{3b},\ \frac{3}{x+y},\ \frac{x+y}{x-y}.$$

From these examples we see that the reciprocal of a fraction is the fraction inverted.

Oral Exercises

Simplify:

1. $\frac{2}{3} \div \frac{4}{9}$.
2. $\frac{1}{x} \div \frac{2}{x}$.
3. $\frac{1}{x^2} \div \frac{1}{x}$.
4. $\frac{2}{x-y} \div \frac{2}{(x-y)^2}$.
5. $\frac{3}{x^3} \div \frac{1}{x^4}$.
6. $\frac{x}{3} \div \frac{x^2}{9}$.
7. $\frac{x^4}{y^5} \div \frac{x^3}{y^2}$.
8. $\frac{xy^5}{8z} \div \frac{x^5y}{yz^2}$.
9. $3x \div \frac{1}{4}$.
10. $\frac{x}{y} \div \frac{1}{4}$.
11. $y^2 \div \frac{y}{a}$.
12. $\frac{4x}{5} \div \frac{8x^2}{10}$.

Written Exercises

Simplify:

1. $\frac{16x^2}{5y^3} \div \frac{8x}{25y^2}$.
2. $\frac{125x^3y^3}{9ab^2} \div \frac{15xy^2}{27a^2}$.
3. $\frac{27m^2ab}{4x^2} \div \frac{6acm}{8x}$.
4. $\frac{6x^3}{4y^2} \div \frac{9x}{5y^3}$.
5. $\frac{12x^2}{9ab} \cdot \frac{18a^3}{4x} \div \frac{6ax}{b}$.
6. $\frac{9a^2b}{8c^2x} \cdot \frac{28ax^2}{15b^2c} \div \frac{21a^3x}{10bc^3}$.
7. $\frac{H+2}{K^2-1} \div \frac{H+2}{K-1}$.
8. $\frac{H^2-4}{3} \div \frac{H-2}{9}$.
9. $\frac{x^2-9}{3x^2+12} \div \frac{x+3}{x^2+4}$.
10. $\frac{x^2-y^2}{x^4-y^4} \div \frac{x-y}{x^2+y^2}$.

11. $\dfrac{a^2 - 4x^2}{a^2 + 4ax} \div \dfrac{a^2 - 2ax}{ax + 4x^2}$.

12. $\dfrac{6ab - 6b^2}{a(a+b)} \div \dfrac{2b^2}{a^3 - ab^2}$.

13. $\dfrac{a^2 - x^2}{a^2 + x^2} \div \dfrac{a - x}{3a^2 + 3x^2}$.

14. $\dfrac{x^2 - 9}{x^2 + x} \div \dfrac{x - 3}{x^2 - 1}$.

15. $\dfrac{(m^2 - 1)^2}{m^3} \div (m^2 - 1)$.

16. $\dfrac{6x^2y - 4xy^2}{45x^2 - 20y^2} \cdot \dfrac{30x + 20y}{4x^2y^2} \div \dfrac{x + y}{xy}$.

17. $\dfrac{x^2 + y^2}{x^2 - y^2} \div \dfrac{3x^2 + 3y^2}{x + y}$.

18. $\dfrac{x^2 - 4y^2}{x^2 + 4xy} \div \dfrac{x^2 - 2xy}{xy + 4y^2}$. Check with $x = 10$ and $y = 4$.

19. Give the reciprocals of $x^2 + 2x + 1$, $a + b + c$.

20. Give the reciprocals of each fraction in example 16.

Optional Exercises

Perform the indicated operations.

21. $\dfrac{2a^2 - 13a + 15}{4a^2 - 9} \cdot \dfrac{2a + 1}{2a - 1} \cdot \dfrac{2a - 1}{a - 5}$.

22. $\dfrac{a^2 - 11a + 30}{a^3 - 6a^2 + 9a} \cdot \dfrac{a^2 - 3a}{a^2 - 25} \cdot \dfrac{a^2 + 2a - 15}{a^2 - 6a}$.

23. $\dfrac{a^2 + 4ab + 3b^2}{a^2 - b^2} \cdot \dfrac{(a - b)^2}{a + 3b} \cdot \dfrac{a + b}{a - b}$.

24. $\dfrac{2a^2 - 5ax - 3x^2}{x^2 - 9a^2} \cdot \dfrac{3x + 9a}{10a^2 + 5ax}$.

25. $\dfrac{x^2 - y^2}{2xy} \cdot \dfrac{x^2(x - 2y)}{3xy - x^2 - 2y^2}$.

26. $\dfrac{2H^2 + 5H + 2}{H^2 - 4} \times \dfrac{H^2 + 4H}{2H^2 + 9H + 4}$.

27. $\dfrac{a^2 + 6a - 7}{a^2 + 15a + 56} \cdot \dfrac{a - 5}{a - 1} \cdot \dfrac{a^2 + 18a + 80}{a^2 + 5a - 50}$.

28. $\dfrac{4a^2-5a-6}{2a^2+3a-2}\cdot\dfrac{12a^2+5a-2}{8a^2-6a-9}\cdot\dfrac{2a^2+a-6}{3a^2-4a-4}$.

29. Does $\dfrac{x^2+x-2}{x^2-7x}\cdot\dfrac{x^2-13x+42}{x^2+2x}=\dfrac{x^2-7x+6}{x^2}$?

Prove it when $x=8$.

30. Find the value of $\dfrac{a^2+2ab}{a^2+4b^2}\cdot\dfrac{ab-2b^2}{a^3-4ab^2}$, and prove your answer when $a=4$, $b=3$.

31. $\dfrac{x^2-4x+3}{3-x}\div\dfrac{x-1}{2-x}$.

32. $\dfrac{c^2-cx-6x^2}{c^3-9cx^2}\div\dfrac{c+2x}{c+3x}$.

33. $\dfrac{x^2-8x+15}{6-5x+x^2}\div\dfrac{7x-10-x^2}{x^2-4x+4}$.

34. $\dfrac{3x^2+x-2}{4x^2-4x-3}\div\dfrac{2x^2-x-3}{6x^2-x-2}$.

35. $\dfrac{x^2+14x-15}{x^2+4x-5}\div\dfrac{x^2+12x-45}{x^2+6x-27}$.

36. $\dfrac{2x^2-x-1}{2x^2+x-1}\div\dfrac{x^2-1}{4x^2-1}$.

37. $\dfrac{y^2+4y+3}{y^2-8y+7}\cdot\dfrac{y^2-2y-35}{y^2-7y-8}\div\dfrac{y^2+8y+15}{y^2-9y+8}$.

38. $\dfrac{x^2+xy}{x^2+y^2}\div\dfrac{x^3y+xy^3+2x^2y^2}{x^4-y^4}$.

39. $\dfrac{y^2-y-12}{y^2-9}\cdot\dfrac{y^2-2y-3}{y^2-2y-8}\div\dfrac{y^2+y}{y+2}$.

88. Lowest Common Multiple.* — What is the smallest number which can be divided by 4, 10, and 15 without a

* Euclid was one of the first mathematicians who knew the method of finding the lowest common multiple of two or more numbers. He was a famous Greek born about 300 B.C. at Alexandria. His Geometry was the standard for thousands of years.

remainder? Answer the same question for 2, 8, and 7; for 3, 5, 4, and 9.

The *lowest common multiple* (*L. C. M.*) of two or more numbers is the smallest number which can be divided by each of those numbers without a remainder.

Thus, 18 is the L. C. M. of 2, 3, and 9. Similarly, the L. C. M. of a, ab, $(x + y)$ is $ab(x + y)$.

We may obtain the L. C. M. of two or more algebraic expressions by the following

Rule. — (1) *Factor each expression into its prime factors.*

(2) *Take each factor the greatest number of times it is found in any one expression.*

This product of these different factors is the lowest common multiple of the expressions.

Illustrative example:

Find the L. C. M. of $ax^2 - axy$, $x^2 - y^2$, and $x^2 + 2xy + y^2$.

Solution:

$$ax^2 - axy = ax(x - y).$$
$$x^2 - y^2 = (x - y)(x + y).$$
$$x^2 + 2xy + y^2 = (x + y)(x + y).$$

The L. C. M., therefore, is $ax(x - y)(x + y)^2$.

The factors a, x, and $(x - y)$ are taken once each because they occur only once in any expression, while $(x + y)$ is taken twice because it occurs twice in the third expression.

Exercises

Find the lowest common multiple of each of the following sets of expressions:

1. 2, 3, 4.
2. 6, 8, 24.
3. $2x$, $5x$.
4. $5x^2$, $6x$.
5. $3x$, x, 13.
6. $4x^2y$, $5xy$, $2x^2y^2$.
7. $3x^2y$, $8x^3yz^2$, $12xz^2$.
8. ab^2, $3a^2b$, $5a^3b^3$.

9. x, y, z, w.
10. $6x - 12, 12x$.
11. $x + y, x - y$.
12. $x^2 - y^2, x + y$.
13. $x, (x - y), y$.
14. $(x + y)^2, x + y$.
15. $3xy + 3y^2, x^2 - y^2$.
16. $x^2 + xy, xy + y^2$.
17. $a, a - b, a^2 + ab$.
18. $a, (a - b)^2, b, (a - b)^3$.
19. $12, 15, 3(a - b), 4(a - b)^2$.
20. $x^2 - xy, 2x - 2y, 2x^2 - 2xy$.
21. $3x^2 - 27, 4x - 12, 4x + 12$.

Find the lowest common multiple of the denominators of each of the following sets of fractions:

22. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}$.
23. $\frac{15}{16}, \frac{17}{18}, \frac{3}{20}$.
24. $\frac{3x}{8}, \frac{7x}{12}, \frac{5x}{3}$.
25. $\frac{3}{x^2 - 1}, \frac{5}{x + 1}$.
26. $\frac{m}{n^2x}, \frac{mn}{3x^2y}, \frac{x^2}{4nxy^2}$.
27. $\frac{3}{x - 1}, \frac{4x}{x^2 - 1}, \frac{x - 2}{5x - 5}$.
28. $\frac{5}{2 - x}, \frac{6}{4 - x^2}, \frac{8x}{6 + 3x}$.
29. $\frac{a}{n^2(n + m)}, \frac{ab}{n(n + m)^2}, \frac{a^2b}{nm^2(n + m)}$.

90. Addition of Fractions. — In section 86 we saw that the value of a fraction is not changed if both numerator and denominator are divided by the same number. In the same way we may multiply both numerator and denominator by the same number without affecting the value of the fraction.

Illustrative example I:

Add $\frac{2}{3}$ and $\frac{1}{2}$.

Solution: $\frac{2}{3} = \frac{4}{6}$ By multiplying both numerator and denominator by 2.

and $\frac{1}{2} = \frac{3}{6}$ Multiplying numerator and denominator by 3.

We now add $\frac{4}{6}$ and $\frac{3}{6}$ by merely adding the numerators. Or we might arrange this as follows:

$$\frac{2}{3} + \frac{1}{2} = \frac{4+3}{6} = \frac{7}{6}.$$

Therefore, $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$.

From the above illustration we see that before fractions can be added they must be changed in form so that they will have the same denominator. This denominator should be the lowest common multiple of the denominators.

Illustrative example II:

Add $\frac{3}{x}$, $\frac{5}{4x}$, and $\frac{7}{2x^2}$.

Solution: The lowest common multiple of the denominators is $4x^2$.

In $\frac{3}{x}$ the denominator must be multiplied by $4x$ (obtained by dividing $4x^2$ by x), and therefore the numerator must be multiplied by $4x$. The fraction then becomes $\frac{12x}{4x^2}$.

Similarly, $\frac{5}{4x} = \frac{5x}{4x^2}$.

And, $\frac{7}{2x^2} = \frac{14}{4x^2}$.

In practice the work is arranged horizontally and the new denominator written but once, as follows:

$$\frac{3}{x} + \frac{5}{4x} + \frac{7}{2x^2} = \frac{12x + 5x + 14}{4x^2} = \frac{17x + 14}{4x^2}.$$

91. Subtraction of Fractions is not to be regarded as a distinct process, but merely as a special case of addition, since

$$\frac{2}{3} - \frac{1}{2} \text{ is the same as } \frac{2}{3} + \left(-\frac{1}{2}\right).$$

Subtracting a positive fraction is, then, merely adding the corresponding negative fraction. We shall usually use the word " combine " or " collect " to imply that the fractions are to be combined into one by algebraic addition. We subtract fractions according to the following

Rule. — (1) *Find the lowest common multiple of the denominators and write it as the denominator of the sum.*

(2) *Divide this new denominator by each of the given denominators in succession and multiply the quotient in each case by the corresponding numerator.*

(3) *Write these products in order over the new denominator, connecting them by the signs of the corresponding fractions.*

(4) *Remove parentheses from the numerator and collect similar terms.*

(5) *If necessary, reduce the result to lowest terms.*

(6) *Check.*

Illustrative example:

Collect $\frac{2}{3\,x} + \frac{5}{6\,x^2} - \frac{3}{12\,x^3}$.

Solution: The common denominator is $12\,x^3$.

$$\frac{2}{3\,x} + \frac{5}{6\,x^2} - \frac{3}{12\,x^3} = \frac{8\,x^2}{12\,x^3} + \frac{10\,x}{12\,x^3} - \frac{3}{12\,x^3} = \frac{8\,x^2 + 10\,x - 3}{12\,x^3}.$$

Checks. — The method of checking addition of fractions may be illustrated by the following:

$$\frac{8}{3x}+\frac{9}{x-1}-\frac{8}{x+1}=\frac{8(x^2-1)}{3x(x^2-1)}+\frac{27x(x+1)}{3x(x^2-1)}-\frac{24x(x-1)}{3x(x^2-1)}$$

$$=\frac{8(x^2-1)+27x(x+1)-24x(x-1)}{3x(x^2-1)}$$

$$=\frac{11x^2+51x-8}{3x(x^2-1)}.$$

Check: 1st in the original example, letting $x = 2$.

$$\frac{8}{3x}+\frac{9}{x-1}-\frac{8}{x+1}=\frac{8}{6}+9-\frac{8}{3}=\frac{46}{6}=7\tfrac{2}{3}.$$

2nd in the result: $$\frac{44+102-8}{6(4-1)}=\frac{138}{18}=7\tfrac{2}{3}.$$

The solution is, therefore, correct.

In checking, you should not substitute 0, or any number which will make any denominator 0.

You should follow each step of the work, noting the application of the rule in detail. For example, when we divide $3x(x^2-1)$ by the first denominator $3x$, we obtain x^2-1. Now multiplying this result by the numerator 8, we obtain $\frac{8(x^2-1)}{3x(x^2-1)}$ as the first fraction.

In a similar manner, each of the other fractions is changed so that its denominator becomes $3x(x^2-1)$.

When we come to combine these numerators, notice that a minus $(-)$ sign before a fraction results in a minus $(-)$ sign before a parenthesis, and watch your signs accordingly.

NOTE. The *lowest common multiple of the denominators* is generally called the *lowest common denominator.*

Oral Exercise

Add or collect:

1. $\frac{2}{5}+\frac{1}{5}$.
2. $\frac{3}{8}+\frac{5}{8}$.
3. $\frac{2}{11}+\frac{7}{11}$.

4. $\frac{1}{2}+\frac{1}{4}$.

5. $\frac{3}{4}+\frac{1}{8}$.

6. $\frac{1}{a}+\frac{3}{a}$.

7. $\frac{2}{5\,b}+\frac{1}{5\,b}$.

8. $\frac{3}{2\,m}-\frac{1}{2\,m}$.

9. $\frac{5}{6\,x}-\frac{4}{6\,x}$.

10. $\frac{3}{10\,m}-\frac{1}{5\,m}$.

11. $\frac{4\,a}{2\,b}+\frac{3\,a}{b}$.

12. $\frac{x}{3\,a}+\frac{3\,x}{a}$.

13. $\frac{2\,m}{s}+\frac{m}{3\,s}$.

14. $\frac{x-y}{4}+\frac{x+y}{4}$.

Exercises

Find the algebraic sum of:

1. $\frac{3}{8}+\frac{5}{16}+\frac{3}{4}$.

2. $\frac{24}{7}+\frac{11}{14}-\frac{21}{2}$.

3. $\frac{7}{3}-\frac{1}{6}-\frac{5}{9}+\frac{7}{2}$.

4. $\frac{9}{32}+\frac{5}{24}+\frac{3}{8}$.

5. $\frac{7}{40}-\frac{11}{60}+\frac{17}{30}-\frac{4}{15}$.

6. $\frac{5}{18}+\frac{25}{36}-\frac{2}{9}-\frac{7}{3}$.

7. $\frac{9}{5}-\frac{7}{10}+\frac{13}{2}$.

8. $\frac{3}{x}+\frac{5}{2\,x}-\frac{11}{3\,x}$.

9. $\frac{5}{4\,a}-\frac{6}{a}-\frac{3}{5\,a}$.

10. $\frac{2}{a^2}+\frac{3}{a}-\frac{5}{2\,a}+\frac{1}{2\,a^2}$.

11. $\frac{4}{3\,x}+\frac{5}{x}-\frac{2}{x^2}$.

12. $\frac{10}{3\,a}-\frac{5}{6\,b}+\frac{10}{9\,ab}$.

13. $\frac{2\,a}{5\,b}+\frac{3\,a}{b}-\frac{3\,a}{10\,b}$.

14. $\frac{2\,a}{c}-\frac{1}{bc}+\frac{a}{2\,c^2}$.

15. $\frac{4\,b}{9\,c^2}+\frac{b}{3\,c}+\frac{4\,b}{9\,c}$.

16. $\frac{x+y}{2}+\frac{x-y}{3}$.

17. $\frac{a+b}{4}-\frac{a-b}{6}$.

18. $\frac{2\,a+3\,b}{3}+\frac{3\,a-2\,b}{5}$.

19. $\frac{m+2\,n}{6}-\frac{m+5\,n}{3}$.

20. $\frac{4\,x-3\,y}{4}-\frac{2\,x+7\,y}{3}$.

21. $\frac{m-n}{10}-\frac{m-n}{15}$.

22. $\frac{4a+3b}{2c}+\frac{a-6b}{3c}$.

23. $\frac{5x-10y+3}{2x}-\frac{x+y}{5x}$.

24. $\frac{m-4n+2}{2m}+\frac{4m+3n-1}{7m}$.

25. $\frac{a+b}{ab}-\frac{a-c}{bc}$.

26. $\frac{4x-3y}{2xy^2}+\frac{x+4y}{3x^2y}$.

27. $\frac{x^2-2y}{2x^2y}-\frac{y+x^2}{3y^2}$.

28. $\frac{3a-2}{2a^2}+\frac{a-3}{2a}+\frac{7}{2}$.

29. $\frac{a-3}{a^2}-\frac{2a+5}{3a^2}+\frac{5a}{6}$.

30. $\frac{b}{x}+\frac{3b^2}{2x}-\frac{b^2+1}{3x}$.

31. $\frac{7a}{2y}-\frac{a}{10y}+\frac{3a-1}{5y^2}$.

32. $\frac{6}{a}+\frac{3}{a-b}-\frac{6b}{a(a-b)}$.

33. $\frac{5}{x+y}-\frac{3}{x-y}$.

34. $\frac{2a}{a-2b}+\frac{2b}{a+2b}$.

35. $\frac{3}{x^2-y^2}-\frac{2}{x+y}$.

36. $\frac{5x}{x^2-4y^2}+\frac{4}{x-2y}$.

37. $\frac{4x^2-5}{3x^2}-\frac{2-3x}{2x}+\frac{3x-7}{5x^3}$.

38. $\frac{a}{a^2-b^2}+\frac{3}{a+b}-\frac{2}{a-b}$.

39. $\frac{5}{a^2-b^2}+\frac{3}{a-b}+\frac{6a}{a+b}$.

40. $\frac{5a+1}{a+3}-\frac{7}{2a}+\frac{2}{3}$.

41. $\frac{4a}{2a+b}-\frac{3b}{a-2b}$.

42. $\frac{3x}{x-3}+\frac{2x}{x+1}$.

43. $\frac{5xy}{y+1}-\frac{2y}{y+2}+\frac{3}{y^2+3y+2}$.

44. $\frac{6a}{a-3b}+\frac{4b}{2a-b}-\frac{2ab}{2a^2-7ab+3b^2}$.

45. $\frac{3}{1-2a}-\frac{7}{1+2a}+\frac{4-20a}{1-4a^2}$.

92. Mixed Expressions. — A *mixed expression* is one in which some of the terms are integral while others are fractional, such as

$$3x - 1 + \frac{3x}{4}.$$

In such cases the integral terms should be written as a fraction having the denominator 1.

Thus,
$$3x - 1 + \frac{3x}{4} = \frac{3x - 1}{1} + \frac{3x}{4}$$
$$= \frac{12x - 4 + 3x}{4}$$
$$= \frac{15x - 4}{4}.$$

Mixed expressions in algebra differ from those of arithmetic in requiring the + sign before the fraction. $2\frac{1}{2}$ means $2 + \frac{1}{2}$, but $2\frac{a}{b}$ means $2 \times \frac{a}{b}$.

Oral Exercises

Change each of the following mixed expressions to a fraction:

1. $4 + \frac{3}{4}$.
2. $5 - \frac{1}{2}$.
3. $\frac{3}{4} + 3 + \frac{5}{6}$.
4. $\frac{1}{2} + x$.
5. $x - \frac{1}{3}$.
6. $2x + \frac{2}{3} - \frac{x}{4}$.
7. $5x + \frac{3}{4} - \frac{x}{2}$.
8. $x + \frac{x}{2} + \frac{x}{3}$.
9. $\frac{5x}{6} - \frac{2x}{3} + x$.
10. $x - 1 + \frac{x}{2}$.
11. $\frac{x + 1}{2} + x - 1$.
12. $b - \frac{b}{x}$.

13. $a - \frac{x}{a}$.

14. $x - \frac{x}{2}$.

15. $x^2 - \frac{1}{2}$.

16. $x^2 - \left(\frac{x}{2}\right)^2$.

17. $\frac{x^2}{6} + x^2$.

18. $\left(\frac{x}{2}\right)^2 - x^2$.

19. $\left(\frac{x}{2}\right)^2 - \frac{-x^2}{4}$.

20. $\left(\frac{x}{6}\right)^2 + \frac{x^2}{6}$.

21. $3 - \frac{2}{x+2}$.

22. $x - \frac{2}{x-1}$.

Written Exercises

Change each mixed expression to a single fraction.

1. $2 + \frac{2}{3}$.

2. $a + \frac{1}{a}$.

3. $4x - \frac{x}{3}$.

4. $6x^2 - \frac{2x}{3}$.

5. $2xy + 3 + \frac{y}{4}$.

6. $\frac{10}{3x} - 5$.

7. $\frac{B^2}{4A} - AC$.

8. $\pi - \frac{\pi r^2}{R^2}$.

9. $\frac{2x}{3} + \frac{5x}{2} - 1$.

10. $\frac{4x}{5} - \frac{3x}{2} + 4$.

11. $\frac{x-1}{3} + 2x - 1$.

12. $\frac{4x+2}{5} - x + 3$.

13. $3a + 2 + \frac{a-1}{a}$.

14. $x^2 - \frac{3x^2}{5}$.

15. $\frac{x^2}{3} + x^3 - \frac{5x^3}{4}$.

16. $5x + 3 - \frac{2x}{x-1}$.

17. $c - d + \frac{c^2+d^2}{c+d}$.

18. $x + \frac{1-x}{2x-1}$.

19. $3 + \frac{a}{a-3} + 5a.$

20. $x - \frac{x^2+y^2}{x+y} + y.$

21. $a^2 - 5a + 6 + \frac{3a^2-2}{a-1}.$

22. $x^2 - xy + y^2 - \frac{y^3}{x+y}.$

93. Change of Signs in Fractions. — Let us consider the fraction $+\frac{+10}{+5}$. Its numerical value is $+2$. We may change any two of the above $+$ signs to $-$ signs without changing the value of the expression.

For,
$$+\frac{-10}{-5} = +(+2) = +2.$$
$$-\frac{+10}{-5} = -(-2) = +2.$$
$$-\frac{-10}{+5} = -(-2) = +2.$$

From these examples we get the following

Principle. — *Of the three signs, the sign of the numerator, the sign of the denominator, and the sign before the fraction, any two may be changed without changing the value of the fraction.*

Exercises

Change the sign of the denominator in each of the following fractions, and balance it by changing the sign of the fraction *or* the sign of the numerator:

1. $\frac{3}{-5}.$

2. $\frac{-2a}{-x}.$

3. $-\frac{x}{-2}.$

4. $\frac{-a}{a-b}.$

5. $\frac{ab}{b-a}.$

6. $-\frac{x^2}{y^2-x^2}.$

7. $\frac{x-1}{2-x}.$

8. $-\frac{x-3}{5-x}.$

9. $-\frac{-2x}{x-y}.$

10. $\frac{-a+b}{a+b}.$

11. $\frac{-2m+n}{n+2m}.$

12. $\frac{n^2-mn}{n^2-m^2}.$

By applying this principle we can sometimes simplify the process of obtaining the lowest common denominator.

Illustrative example:

Combine $\dfrac{5x}{x^2-1}+\dfrac{3x}{1-x}$.

The factors of the first denominator are $(x+1)(x-1)$. This would be the lowest common denominator if the denominator of the second fraction were $x-1$ instead of $1-x$. $1-x$ can be changed to $x-1$ by changing the sign before the fraction by the above principle. We now have

$$\frac{5x}{x^2-1}-\frac{3x}{x-1}=\frac{5x}{x^2-1}-\frac{3x(x+1)}{x^2-1}$$
$$=\frac{5x-3x(x+1)}{x^2-1}$$
$$=\frac{5x-3x^2-3x}{x^2-1}$$
$$=\frac{2x-3x^2}{x^2-1}.$$

Exercises

Add algebraically:

1. $\dfrac{4x}{x^2-4}+\dfrac{3x-2}{4-x^2}$.
2. $\dfrac{4}{3-x}+\dfrac{2}{x-3}$.
3. $\dfrac{2a}{a+1}+\dfrac{3}{a^2-a}$.
4. $\dfrac{5a}{a+5}+\dfrac{3}{25-a^2}$.
5. $\dfrac{x^2-2y^2}{x^2-y^2}-\dfrac{3y}{y-x}$.
6. $\dfrac{H}{1-H^2}-\dfrac{1}{H-1}$.
7. $\dfrac{b^2}{b^2-1}+\dfrac{b}{b+1}-\dfrac{b}{1-b}$.
8. $\dfrac{5x}{x^2-16}-\dfrac{3}{x-4}+\dfrac{2}{4-x}$.
9. $\dfrac{4x}{x-7}-\dfrac{3x}{x+7}-\dfrac{2}{49-x^2}$.
10. $\dfrac{a-3}{a-6}-\dfrac{5}{6-a}+2a$.
11. $\dfrac{y^2}{y^2-9}+\dfrac{2y}{3-y}+4y^2$.
12. $5a+3+\dfrac{a^2}{a+1}+\dfrac{2a}{1-a}$.

13. $\dfrac{5}{1+a}+\dfrac{a}{1-a^2}+\dfrac{2\,a}{a^2-1}$.

14. $\dfrac{3+2\,a}{2-a}+\dfrac{16\,a-a^2}{a^2-4}-\dfrac{2-3\,a}{2+a}$.

15. $\dfrac{a}{a+1}+2-\dfrac{1}{a-1}$.

16. $\dfrac{2-2\,x}{(x-1)^3}+\dfrac{x+1}{(x-1)^2}+\dfrac{1}{1-x}$.

17. $\dfrac{x}{x-y}+\dfrac{2\,xy}{y^2-x^2}-\dfrac{2\,x}{x+y}$.

18. $\dfrac{8}{x+3}+\dfrac{24}{9-x^2}$.

19. $\dfrac{3\,b^2}{a^2-b^2}+\dfrac{2\,a^2}{b^2+2\,ab-3\,a^2}$.

20. $\dfrac{y}{y^2-y-2}+\dfrac{2\,y+1}{6-3\,y}+\dfrac{3}{8}$.

21. $\dfrac{a+b}{a^2+2\,ab+b^2}+\dfrac{a-b}{a^2-b^2}-\dfrac{a-b}{a^2-2\,ab+b^2}$.

Optional

Reduce each of the following fractions to its lowest terms and combine the results:

22. $\dfrac{a-2}{a^2-4}-\dfrac{2}{a-5}$.

23. $\dfrac{a+ax}{(1+x)^2}-4-\dfrac{a+ax}{1-x^2}$.

24. $\dfrac{2\,\pi R}{2\,\pi R^2-2\,\pi r^2}-\dfrac{2\,\pi R}{2\,\pi R^2-2\,\pi H}$.

25. Does $\dfrac{R}{1+R}-\dfrac{R}{1-R}+\dfrac{R^2}{R^2-1}=\dfrac{3\,R^2}{1-R^2}$ if $R=4$?

Selection Test

Select the quantity in the parenthesis which correctly supplies the missing part:

1. $\frac{1}{2} + \frac{1}{3} = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{6}, \frac{2}{6}, \frac{5}{6}\right)$.

2. $4 \div \frac{1}{2} = (2, 8, 6)$.

3. $\frac{a}{x} + \frac{y}{c} = \left(ac + xy, \frac{ac + xy}{cx}, \frac{a + y}{x + c}\right)$.

4. If $K = \frac{1}{2}bc$, $b = \left(\frac{\frac{1}{2}K}{c}, \frac{2K}{c}, \frac{2c}{K}, \frac{\frac{1}{2}c}{K}\right)$.

5. $\frac{2a + 4}{6a + 4} = \left(\frac{a + 4}{3a + 4}, \frac{a}{3}, \frac{a + 2}{3a + 2}, \frac{1}{3}\right)$.

6. $\frac{a}{c} \cdot \frac{c}{a} = (1, 0)$.

7. $\frac{a}{b} - \frac{a}{b} = (1, 0)$.

8. The L.C.D. of $\frac{2a}{a - b} + \frac{a}{2a - b} - \frac{3}{a}$ is $[a(a - b)(2a - b),\ a(2a - b),\ 2a - b, 4a - 2b]$.

9. $\dfrac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{c} - \frac{1}{d}} = \left(\frac{a + b}{d - c}, \frac{(a + b)(c - d)}{abcd}, \frac{cd(a + b)}{ab(d - c)}\right)$.

10. $\frac{a - b}{ab} - \frac{1}{b} + \frac{1}{a} = \left(0, \frac{1}{ab}, 1\right)$.

94. Multiplications and Divisions Involving Mixed Expressions. — Sometimes we must perform a simple addition or subtraction before we can multiply or divide. This is shown by the following

Illustrative example I:

Multiply $\left(\frac{x}{a} - \frac{a}{x}\right) \cdot \frac{a}{x+a}$.

Solution:

$$\frac{x}{a} - \frac{a}{x} = \frac{x^2 - a^2}{ax}.$$

$$\frac{(x+a)(x-a)}{ax} \cdot \frac{a}{a+x} = \frac{x-a}{x}.$$

Illustrative example II:

Simplify $\frac{7-2a}{9-a^2} \div \left(2 + \frac{1}{3-a}\right)$.

Solution:

$$2 + \frac{1}{3-a} = \frac{6-2a+1}{3-a} = \frac{7-2a}{3-a}.$$

$$\frac{7-2a}{9-a^2} \div \frac{7-2a}{3-a} = \frac{1}{(3+a)(3-a)} \cdot \frac{1}{1} \cdot \frac{3-a}{1} = \frac{1}{3+a}.$$

Check: Put 2 for a in answer and in example.

Exercises

Simplify each of the following:

1. $\left(x + 2 + \frac{1}{x}\right) \cdot \frac{3x}{x^2-1}$.

2. $\frac{a-b}{a+b}\left(1 + \frac{b}{a}\right)$.

3. $\left(1 - \frac{b^2}{a^2}\right)\left(3 - \frac{3b}{a+b}\right)$.

4. $\left(4 + \frac{3}{m^2-1}\right)\left(\frac{3m}{2m-1} - 1\right)$.

5. $\left(2 - \frac{c}{2}\right)^2 \div \frac{4-c}{4}$.

6. $\left(\frac{m}{3} - \frac{3}{m}\right) \div \frac{m^2 - 6m + 9}{3m}$.

7. $\left(3a - \frac{2ab + b^2}{a}\right) \div \frac{a^2 - ab}{a^2b^2}$.

8. $\left(\frac{x^2}{6} - 6\right) \div \left(1 + \frac{x}{6}\right)$.

9. $\left(a - \frac{a^2}{b}\right) \div \left(1 - \frac{a}{b}\right)$.

10. $\left(1 + \frac{1}{x-1}\right) \div \left(1 - \frac{1}{x+1}\right)$.

11. $\left(\frac{1}{a} + \frac{2}{a+b}\right) \div \left(\frac{1}{a} + \frac{1}{b}\right)$.

12. $\left(4 - \frac{6}{a+1}\right) \div \left(8 - \frac{4a-8}{a^2-1}\right)$.

13. $\left(\frac{x+y}{x} + \frac{x-y}{y}\right)\left(\frac{x-y}{x} - \frac{x+y}{y}\right)\frac{x^2y^2}{x^2+y^2}$.

14. $\left(\frac{a^2+ax}{2x}\right)\left(\frac{1}{a^2-x^2}\right)\left(\frac{(a+x)^2}{4ax} - 1\right)$.

15. $\left(1 - \frac{1-x}{1+x} + \frac{1+2x^2}{1-x^2}\right)\left(\frac{x+1}{2x+1}\right)$.

16. $\frac{H}{2}\left(\frac{1}{H-K} - \frac{1}{H+K}\right) \cdot \frac{H^2-K^2}{H^2K+HK^2} \div \frac{1}{H+K}$.

17. $\frac{x^2-(a-1)^2}{a^2-(x+1)^2} \cdot \frac{(a+x)^2-1}{1-(a-x)^2} \div \frac{a+x-1}{a-x-1}$.

18. $\left(\frac{1}{x^2-y^2} + \frac{1}{x-y} - \frac{1}{x+y}\right) \div \frac{2x+1}{x-y}$.

19. $\left(\frac{x-y}{x+y} + 1\right) \div \left(\frac{x-y}{x+y} - \frac{x+y}{x-y}\right)$.

20. $\left(\frac{2x+3y}{x-y} - \frac{x-y}{2x+3y}\right) \div \left(\frac{3x+2y}{x-y} \cdot \frac{x+4y}{2x+3y}\right)$.

True-False Test

State whether each of the following statements is true or false:

1. $\frac{1}{x} + \frac{3}{y}$ is the same as $\frac{4}{x+y}$.

2. $\frac{x}{y} + \frac{z}{y}$ is the same as $\frac{x+z}{y}$.

3. $\frac{a}{b} + c$ equals $a + bc$.

4. $\frac{a+b-c}{2}$ is the same as $\frac{a}{2} + \frac{b}{2} - \frac{c}{2}$.

5. $\frac{a}{c} - \frac{b}{d}$ equals $\frac{ad - bc}{cd}$.

6. $\frac{2a+2b}{6}$ may be reduced to $\frac{a+b}{3}$.

7. $\frac{5x-y}{10}$ may be reduced to $\frac{x-y}{2}$.

8. $\frac{x+2}{a+2}$ may be reduced to $\frac{x}{a}$.

9. $\frac{+(a-b)}{-(b-a)} = 1$.

10. $\frac{a-1}{1-a} = -1$.

11. $\frac{a-b}{2c} = -\frac{b-a}{2c}$.

12. $\frac{a-b}{2c} = -\frac{a+b}{2c}$.

13. $\frac{a+b}{2} \cdot \frac{2a}{a^2-b^2} \cdot \frac{a-b}{a} = 0$.

14. $\frac{a+b}{c-d} = -\frac{a+b}{d-c}$.

15. If a and b are positive numbers, $\frac{a}{2b} > \frac{a}{b}$.

Hint: $>$ means "is greater than."

95. Complex Fractions. — We have already seen that division is often represented in algebra by the fractional form rather than by the division sign.

Thus $\left(\frac{1}{a}+\frac{2}{a+b}\right)\div\left(\frac{1}{a}+\frac{1}{b}\right)$ might be written as $\dfrac{\frac{1}{a}+\frac{2}{a+b}}{\frac{1}{a}+\frac{1}{b}}$.

In the latter form it is called a *complex fraction*. The method of indicating the division does not, however, affect the solution of the example.

Solution:

Reducing the fractions in the numerator to a common denominator,

$$\frac{1}{a}+\frac{2}{a+b}=\frac{a+b+2a}{a(a+b)}=\frac{3a+b}{a(a+b)}.$$

Reducing the fractions in the denominator to a common denominator,

$$\frac{1}{a}+\frac{1}{b}=\frac{b+a}{ab}.$$

Inverting the divisor and multiplying,

$$\frac{3a+b}{a(a+b)}\div\frac{b+a}{ab}=\frac{3a+b}{a(a+b)}\cdot\frac{ab}{a+b}=\frac{b(3a+b)}{(a+b)^2}.$$

Check: Put 2 for a and 3 for b in the original, giving

$$\frac{\frac{1}{2}+\frac{2}{2+3}}{\frac{1}{2}+\frac{1}{3}}=\frac{\frac{1}{2}+\frac{2}{5}}{\frac{1}{2}+\frac{1}{3}}=\frac{\frac{9}{10}}{\frac{5}{6}}=\frac{27}{25}.$$

Now substitute in the answer, giving

$$\frac{3(6+3)}{(2+3)^2}=\frac{27}{25}.$$

Exercises

In like manner simplify the following. Check.

1. $\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}$.

2. $\dfrac{2+\frac{2}{3}}{2-\frac{2}{3}}$.

3. $\dfrac{\frac{1}{2}+\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}$.

4. $\dfrac{\frac{m^2+n^2}{m^2-n^2}}{\frac{2m^2+2n^2}{m-n}}$.

5. $\dfrac{\frac{8}{m}-m^2}{\frac{m}{2}+1+\frac{2}{m}}$.

6. $\dfrac{\frac{x^2}{6}-6}{1+\frac{x}{6}}$.

7. $\dfrac{a-\frac{a^2}{b}}{1-\frac{a}{b}}$.

8. $\dfrac{1+\frac{1}{x-1}}{1-\frac{1}{x+1}}$.

9. $\dfrac{a-\frac{ab}{a+b}}{a+\frac{ab}{a-b}}$.

10. $\dfrac{\frac{4}{x+1}-\frac{2}{x+2}}{\frac{(x+1)^2-4}{x^2+x-2}}$.

11. $\dfrac{\frac{a^2}{x}-\frac{x^2}{a}}{x+\frac{x^2}{a}+a}$.

12. $\dfrac{1+\left(\frac{1}{x}\right)^2}{\frac{1}{x^2}}$.

13. $\dfrac{1+\left(-\frac{y}{x}\right)^2}{\frac{2y}{x^2}}$.

14. $\dfrac{1+\frac{1}{a-1}}{1-\frac{1}{a-1}}$.

15. $\dfrac{1+\frac{y}{x-y}}{1-\frac{y}{x+y}}$.

16. $\dfrac{\frac{a-b}{a}-\frac{a+b}{b}}{\frac{a-b}{b}+\frac{a+b}{a}}$.

17. $\dfrac{x-\frac{4}{x}}{\frac{1}{x^2}+\frac{2}{x^3}-\frac{8}{x^4}}$.

18. $1-\dfrac{4}{5+\frac{3}{4-\frac{1}{3}}}$.

True-False Test

Some of the following are true and some are false. Copy the numbers of the examples and write a *t* or an *f* after each as you believe them to be true or false.

1. $\frac{7}{8} = \frac{7+3}{8+3}$.

2. $\frac{x}{y} = \frac{x+4}{y+4}$.

3. $\frac{3}{5} = \frac{12}{20}$.

4. $\frac{x}{y} = \frac{4x}{4y}$.

5. $\frac{2}{3} = \frac{2-1}{3-1}$.

6. $\frac{x}{y} = \frac{x-3}{y-3}$.

7. $\frac{x}{6} \cdot 6 = x$.

8. $5(a-4) = 5a - 4$.

9. $y - 8 - 5 = y - 3$.

10. $\frac{x}{3} = \frac{1}{3}$ of x.

11. $x^2 + y^2 = (x+y)^2$.

12. $x^2 - y^2 = (x-y)^2$.

13. $(x-2)^2 = (2-x)^2$.

14. $x - 2 = 2 - x$.

Assume x is not equal to 2.

15. $\frac{x}{y} = \frac{-x}{-y}$.

16. $\frac{3+6}{3} = 1 + 2$.

17. $\frac{3+9}{3} = 1 + 9$.

18. $\frac{x}{5} \div 5 = x$.

19. $\frac{x}{20} \div \frac{1}{5} = \frac{x}{4}$.

20. $\frac{1}{2} \div \frac{1}{3} = \frac{2}{3}$.

21. $\frac{x}{3} \div \frac{x}{5} = 1\frac{2}{3}$.

22. $\frac{x(y+4)}{z(y+4)} = \frac{x}{z}$.

23. $\frac{6(x+2)}{12(x+2)} = \frac{1}{2}$.

24. $\frac{1}{3}(12 + 9 + 6) = 4 + 3 + 2$.

25. $x + x^2 + x^3 = x(x + x^2)$.

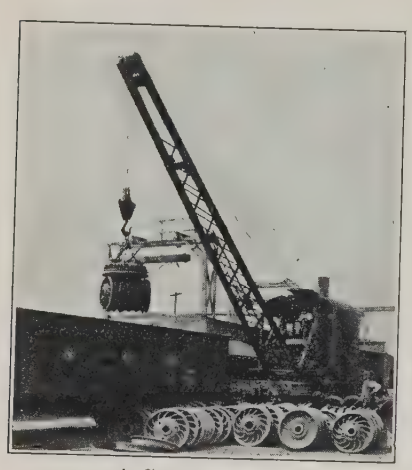
Review

1. Reduce to lowest terms:

$$\frac{2x^3 + x^2 - 6x}{2x^3 - 11x^2 + 12x}.$$

2. A piece of bronze weighs 5401 pounds. It contains twice as much tin as zinc, and 8 times as much copper as zinc. How many pounds of each metal does it contain?

3. Multiply $.4\,x^2 + .4\,x + .5$ by $.4\,x + .6$ and check with $x = 10$.



A Steam Crane

4. The *output* of a steam crane (for lifting) is denoted by the product of the weight lifted (in pounds) by the number of feet the weight is lifted. The *input* may be found by multiplying the number of minutes required to lift the weight by the amount of horse power of the machine. (The horse power is the rate of work and equals 33,000 foot-pounds per minute.) The *efficiency* is denoted by the output divided by the input. Write formulas for *output*, *input*, and *efficiency*.

5. Apply your formulas of number 1 to the following problem:

A steam crane working at 3 horse power raises a block of granite weighing 8 tons, 50 feet in 12 minutes. Find its efficiency.

6. Find two complementary angles (angles whose sum equals 90 degrees) such that twice one of them increased by three times the other equals 210 degrees.

7. Combine: $\dfrac{5}{x^2 - 9} - \dfrac{6}{x - 3} + \dfrac{8}{x + 3}$.

8. Find the value of $\dfrac{a^3 + b^3}{a - b}$ and state your result as a mixed expression.

9. Solve the quadratic equation $2x^2 - 4x - 48 = 0$, by factoring, for the value of x. Check your results.

10. The area of a certain square is equal to the area of a rectangle whose length is 4 feet more and whose width is 3 feet less than a side of the square. Find a side of the square.

11. Find in the shortest way the cost of 82 yards of cloth at 78 cents per yard.

12. What is the value of 8 times 0 plus 6? Of $12 - 9 \cdot 0$? Of $15 \cdot 0 + 14 + 6 \cdot 0$?

13. If a tank empties at the rate of 600 gallons a minute, how many quarts will run out of it in one third of an hour? In x minutes? In $x - 6$ minutes?

14. Will $\frac{y+3}{4} + \frac{y-6}{y} = \frac{y^2+9y-24}{4y}$ when $y = 4$?

Solve for x and check:

15. $6x - (3x + 7) = 13x - (5x - 8)$.

16. $.6x - 8 = .4x + .6$.

17. $a^2x + b^2x = a^4 + 2a^2b^2 + b^4$.

18. Solve $W = fs$ for s.

19. Write a formula for cost (c) of sending a telegram if the first 10 words cost 50¢ and each additional word (w) costs 2 cents.

20. Graph the wage formula $W = .50h$ for integral values of h from 0 to 8. How does W change as h increases?

21. Make a distribution graph if 5 pupils make 90 to 100; 15 make 80 to 90; 20 make 70 to 80; and 5 are below 70.

22. How do each of the following compare in size with $\frac{x}{y}$ when $x = 3$, $y = 4$,

$$\frac{x}{y-1}, \quad \frac{x}{5y}, \quad \frac{x-2}{y-2}, \quad \frac{4x}{y}, \quad \frac{x+4}{y+4}, \quad \frac{x^3}{y^3}, \quad \frac{6x}{6y}$$

CHAPTER IX

FRACTIONAL EQUATIONS

We have just learned how to perform ordinary operations with fractions. These we are now to apply in solving equations and problems which contain fractions. A large number of problems require a knowledge of fractional equations before they can be solved.

96. Fractional Equations Having Monomial Denominators. — In section 55 we learned the use of axioms in solving equations. One of these, the axiom of multiplication, we use to solve an equation containing fractions.

If both members of an equation are multiplied by the same quantity, the equation remains true.

Illustrative example I:

Solve $\frac{x}{3} + \frac{5x}{9} = x - 1.$

Solution: Here we multiply both members of the equation by 9, the L. C. D. (lowest common denominator), as follows:

$$\frac{x}{\cancel{3}} \cdot \overset{3}{\cancel{9}} + \frac{5x}{\cancel{9}} \cdot \cancel{9} = 9(x - 1),$$

obtaining $3x + 5x = 9x - 9.$

$-x = -9.$ By transposing and combining terms.

$x = 9$, root. By dividing both members by -1.

The purpose in multiplying both members of the equation by 9 is to remove the fractions from the equation. To do

this we have to multiply by some number into which each of the given denominators can be divided evenly, so that the denominators will divide out. The best number to choose for this purpose is the lowest common multiple of the denominators.

We solve equations containing fractions by the following

Rule. — (1) *Remove fractions from the equation by multiplying both members by the lowest common denominator of all the fractions.*

(2) *Solve the resulting simple equation by the rule stated in section 58.*

In applying the multiplication axiom to the solution of fractional equations, we should learn to perform our divisions mentally.

Illustrative example:

Solve the equation $\frac{2}{x} + \frac{16}{15} = \frac{19}{3x}$.

The lowest common multiple of the denominators is $15x$. Without writing $15x$ beside each fraction, we may mentally divide $15x$ by x, obtaining 15. This result must be multiplied by the numerator 2. In like manner we divide $15x$ by 15, obtaining x, which we multiply by 16. Then, dividing $15x$ by $3x$, we obtain 5, which is multiplied by 19. The resulting equation is

$$30 + 16x = 95.$$

Hence

$$16x = 65.$$

$$x = 4\tfrac{1}{16}.$$

This process of mental division is described by the

Rule. — *Divide the lowest common multiple of the denominators by the given denominator, and multiply the result by the numerator.*

Oral Exercises

Solve:

1. $\frac{x}{3} = 2.$
2. $\frac{x}{5} = 6.$
3. $\frac{x}{10} = -1.$
4. $\frac{2x}{3} = 6.$
5. $\frac{3x}{5} = 12.$
6. $\frac{4x}{9} = 8.$
7. $\frac{x}{2} = \frac{3}{4}.$
8. $\frac{x}{3} = \frac{4}{5}.$
9. $\frac{1}{x} = 2.$
10. $5 = \frac{2}{x}.$
11. $\frac{3}{x} = 6.$
12. $\frac{5}{x} = -5.$
13. $\frac{2x}{5} = 10.$
14. $\frac{x}{3} = \frac{5}{6}.$
15. $\frac{2}{x} = 1.$
16. $\frac{x}{3} = \frac{2}{3}.$
17. $4 = \frac{16}{x}.$
18. $0 = \frac{x}{5}.$
19. $\frac{1}{x} = \frac{3}{4}.$
20. $\frac{2}{x} = \frac{6}{7}.$
21. $\frac{2}{x} = -\frac{1}{4}.$
22. $\frac{x}{2} + 6 = 0.$
23. $\frac{2x}{3} - 1 = 7.$
24. $0 = 10 - \frac{5x}{6}.$
25. $\frac{y}{b} = c.$
26. $\frac{6}{5} = \frac{10x}{15}.$
27. $\frac{3x}{4} - 6 = 0.$

Written Exercises

Solve and check as directed by the teacher:

1. $\frac{x}{2} - \frac{x}{3} = 2.$
2. $\frac{x}{3} - \frac{x}{4} = 3.$
3. $x + \frac{x}{2} + \frac{x}{3} = 11.$
4. $y + \frac{y}{2} + \frac{y}{3} + \frac{y}{4} = 25.$
5. $\frac{y}{4} + \frac{1}{2} = -3.$
6. $\frac{x}{4} - \frac{x}{5} = 4.$
7. $3 + \frac{x}{4} = x + 6.$
8. $\frac{x}{4} - 5 = 4 - \frac{x}{5}.$

9. $\frac{1}{2x} + \frac{1}{3x} = \frac{1}{6}$.

10. $\frac{x}{3} - \frac{x}{15} = 14 - \frac{x}{5}$.

11. $\frac{4x}{9} - 7 = 0 - \frac{x}{3}$.

12. $\frac{1}{2} + \frac{3}{4} = x$.

13. $\frac{3}{4} - \frac{1}{2} = x$.

14. $x - \frac{3x}{2} = 7\frac{1}{2}$.

15. $\frac{2}{3}n + \frac{5}{4}n = \frac{23}{24}$.

16. $x = 1 + \frac{x}{2} + \frac{x}{4} + \frac{x}{8} + \frac{x}{16}$.

97. Fractional Equations Having Binomial Numerators.

Illustrative Example:

Solve $\frac{2x}{3} - \frac{x+5}{2} = \frac{3x-3}{4}$.

Solution: The lowest common denominator is 12.

Multiplying every term by 12,

$$\overset{4}{\cancel{12}} \cdot \frac{2x}{\cancel{3}} - \overset{6}{\cancel{12}} \cdot \frac{(x+5)}{\cancel{2}} = \overset{3}{\cancel{12}} \cdot \frac{(3x-3)}{\cancel{4}},$$

we obtain $8x - 6(x+5) = 3(3x-3)$

or $8x - 6x - 30 = 9x - 9$.

Transposing, and combining like terms,

$$-7x = 21.$$

Dividing by -7, $x = -3$, root.

Check. $x = -3$.

$$\frac{-6}{3} - \frac{-3+5}{2} = \frac{-9-3}{4}$$

$$-2 - 1 = -3$$

$$-3 = -3.$$

Notice the effect of the $-$ sign before the second fraction. A $-$ sign before a fraction is equivalent to a $-$ sign before the numerator inclosed in a parenthesis, and the signs in all such cases must be carefully watched.

Exercises

Solve and check:

1. $\frac{x+6}{4} = \frac{9}{2}$.

2. $\frac{x+3}{2} = \frac{27}{9}$.

3. $\frac{5x-3}{2} + 14 = 0$.

4. $\frac{x-7}{2} = \frac{7-x}{5}$.

5. $2x - 8 - \frac{24-2x}{7} = 0$.

6. $\frac{2x-12}{3x} = 2$.

7. $\frac{x+3}{4} + \frac{4x-5}{5} = 5$.

8. $\frac{x-2}{2} + \frac{3x+2}{2} = 6$.

9. $\frac{2x+13}{3} - \frac{6-x}{4} = 1$.

10. $4x + \frac{6x}{7} = \frac{3x+2}{2} + 46$.

11. $\frac{3(x-1)}{4} + \frac{5x-7}{4} = \frac{3}{2}$.

12. $\frac{2n+3}{5} - \frac{n-3}{3} = 2$.

13. $\frac{x}{5} - \frac{x-2}{3} + \frac{x}{2} = \frac{13}{3}$.

14. $\frac{5z+7}{2} + \frac{2z+4}{3} = \frac{3z+9}{4} + 5$.

15. $\frac{y+4}{3} - \frac{y-4}{5} = 2 + \frac{3y-1}{15}$.

16. $\frac{5-2y}{4} + 2 = y - \frac{6y-8}{2}$.

17. $\frac{x+6}{11} - \frac{2x-18}{3} + \frac{2x+3}{4} = 5\frac{1}{3} + \frac{3x+4}{12}$.

18. $\frac{2}{3}(x-4) = -6$.

19. $\frac{3}{4}(x-8) = 9$.

Hint: First remove the parenthesis. $\frac{2}{3}x - \frac{8}{3} = -6$.

20. $\frac{1}{2}(x-4) = \frac{2}{3}(x-5)$.

21. $4x - \frac{1}{6}(3+5x) = 9$.

22. $\frac{1}{5}(x-1) + \frac{1}{4}(x-2) + \frac{1}{3}(x-3) = 3$.

23. $\frac{1}{7}(3y-4) + \frac{1}{3}(5y+3) = 43 - 5y$.

24. $4 - \frac{5y-15}{4} = \frac{5(y-1)}{6} - \frac{2(y+2)}{3}$.

25. $\frac{4x+3}{5} - \left(x - \frac{2x-2}{6}\right) = 4.$

26. $\frac{1}{4}(2x-3) - \frac{1}{6}(x+1) = \frac{1}{12}(5x+2).$

27. $3x - \left(2x - \frac{x+7}{5}\right) = \frac{x+4}{6} + 9.$

28. $\frac{1}{2}(x+\frac{1}{3}) - \frac{1}{3}(x-\frac{1}{4}) = 0.$

Reduce fractions to lowest terms and find the root of the simplified equation:

29. $\frac{y}{4} + \frac{y^2-1}{y+1} = 4.$

30. $\frac{x^2+2x+1}{x+1} - \frac{4x}{6} = 8.$

31. $\frac{x^2-25}{x-5} + \frac{x^2+5x}{x+5} = 23 - 4x.$

Time-Practice Test. $1\frac{1}{2}$ minutes.

Solve each of the following 20 examples:

1. $\frac{1}{2}x = 5,\ x =$

2. $\frac{1}{4}x = 7,\ x =$

3. $\frac{1}{5}x = 6,\ x =$

4. $\frac{x}{8} = 7,\ x =$

5. $4 = \frac{1}{3}x,\ x =$

6. $12 = \frac{1}{6}x,\ x =$

7. $\frac{x}{9} = 20,\ x =$

8. $\frac{x}{10} = 40,\ x =$

9. $50 = \frac{x}{5},\ x =$

10. $\frac{1}{11}x = 24,\ x =$

11. $0.1x = 5,\ x =$

12. $\frac{1}{20}x = 6,\ x =$

13. $\frac{x}{70} = 4,\ x =$

14. $\frac{x}{50} = 6,\ x =$

15. $\frac{x}{80} = 10,\ x =$

16. $\frac{x}{30} = 7,\ x =$

17. $12 = \frac{x}{20}$, $x =$ **19.** $\frac{x}{40} = 5$, $x =$

18. $10 = \frac{x}{25}$, $x =$ **20.** $\frac{x}{90} = 12$, $x =$

98. Fractional Equations Having Binomial Denominators.

Illustrative example I:

Solve and check $\frac{7}{4} - \frac{2x-1}{x+1} = \frac{1}{2}$.

Solution: The L. C. D. is $4(x+1)$. Multiply each term by it.

We obtain $\overset{1}{\cancel{4}}(x+1)\frac{7}{\cancel{4}} - 4\cancel{(x+1)}\frac{2x-1}{\cancel{x+1}} = \overset{2}{\cancel{4}}(x+1)\frac{1}{\cancel{2}}$

$$7(x+1) - 4(2x-1) = 2(x+1)$$
$$7x + 7 - 8x + 4 = 2x + 2$$
$$-3x = -9$$

Dividing by -3, $x = 3$, root.

Check: $x = 3$.

$$\frac{7}{4} - \frac{6-1}{4} = \frac{1}{2}$$
$$\frac{7}{4} - \frac{5}{4} = \frac{1}{2}$$
$$\frac{1}{2} = \frac{1}{2}.$$

Illustrative example II:

Solve $\frac{7x-11}{x-11} - \frac{4x-2}{x+1} = 3$.

Solution: The L. C. M. of the denominators is

$$(x-11)(x+1).$$

Multiplying each term of the equation by $(x - 11)(x + 1)$, we obtain

$(x + 1)(7x - 11) - (x - 11)(4x - 2) = 3(x - 11)(x + 1)$, or

$(7x^2 - 4x - 11) - (4x^2 - 46x + 22) = 3(x^2 - 10x - 11)$, or

$7x^2 - 4x - 11 - 4x^2 + 46x - 22 = 3x^2 - 30x - 33.$

$72x = 0.$ Transposing, and combining similar terms.

Check as in example I. $x = 0.$ Dividing by 72.

Exercises

Solve and check each root:

1. $\dfrac{x}{x-6} = 3.$

2. $\dfrac{x}{x-4} = 5.$

3. $\dfrac{x}{x-7} = 8.$

4. $\dfrac{8x}{2+x} = 4.$

5. $\dfrac{12x}{6+x} = 4.$

6. $\dfrac{x}{2x+6} = \dfrac{3}{5}.$

7. $\dfrac{2n}{n-2} = \dfrac{5}{3}.$

8. $\dfrac{x}{3x-5} = \dfrac{2}{3}.$

9. $\dfrac{2}{x} = \dfrac{4}{3x-5}.$

10. $\dfrac{1}{x} = \dfrac{12}{7x-5}.$

11. $\dfrac{7}{x+3} + \dfrac{3}{5} = 2.$

12. $\dfrac{3x}{2x+3} - \dfrac{1}{5} = 1.$

13. $3 - \dfrac{5}{2x-1} = \dfrac{16}{7}.$

14. $\dfrac{5}{2} - \dfrac{8}{x+5} = -\dfrac{3}{2}.$

15. $\dfrac{11}{2} - \dfrac{15}{6-x} = 3.$

16. $1 + \dfrac{2y}{3y-1} = \dfrac{5}{2}.$

17. $\dfrac{12x^2}{3x+5} = 4x - 5.$

18. $\dfrac{20x^2}{4x-3} = 5x + 3.$

Binomials in Numerator or Denominator or in Both.

19. $\dfrac{x-3}{x+7}=6.$

20. $\dfrac{x+9}{x-3}=3.$

21. $\dfrac{x-10}{x+2}=5.$

22. $\dfrac{x-3}{2x+5}=6.$

23. $\dfrac{x-6}{x+3}=\dfrac{3}{4}.$

24. $\dfrac{x-7}{x+9}=\dfrac{5}{6}.$

25. $\dfrac{6x}{3x+1}=\dfrac{2x-1}{x}.$

26. $\dfrac{2x}{x-3}=\dfrac{2x+1}{x}.$

27. $\dfrac{x}{5-x}=\dfrac{1-4x}{4x-5}.$

28. $\dfrac{x-3}{x+1}=\dfrac{x-6}{x-7}.$

29. $\dfrac{2x-3}{3x-1}=\dfrac{2x+3}{3x+5}$

30. $\dfrac{y+1}{y+7}=\dfrac{y-4}{y-1}.$

31. $\dfrac{x-2}{x+3}=\dfrac{x+6}{x-5}.$

32. $\dfrac{2y-5}{3y-7}=\dfrac{2y-7}{3y-5}.$

33. $\dfrac{6}{x-1}+\dfrac{10}{x+1}=\dfrac{60}{x^2-1}.$

34. $\dfrac{1}{x}+\dfrac{2}{x-1}=\dfrac{8}{x^2-x}.$

35. $\dfrac{2}{x-2}-\dfrac{2}{x^2-4}=\dfrac{5}{x+2}.$

36. $\dfrac{4}{x+2}+\dfrac{x+1}{x-2}=\dfrac{x^2-3}{x^2-4}.$

99. Fractional Equations Using Change of Sign.

Illustrative example:

Solve $\dfrac{3}{x-3}-\dfrac{4}{3+x}+\dfrac{8x+3}{9-x^2}=0.$

Solution: We may change the sign of the first denominator, and balance it by placing a $-$ sign before the fraction, as explained in section 93. We then have

$$-\frac{3}{3-x}-\frac{4}{3+x}+\frac{8x+3}{(3+x)(3-x)}=0.$$

The L. C. M. of the denominators is $(3+x)(3-x)$.

Multiplying each term of the equation by this quantity we obtain

$$-3(3+x)-4(3-x)+8x+3=0.$$
$$-9-3x-12+4x+8x+3=0.$$

Hence, $9x=18$,

and $x=2$, root.

Check as usual.

Exercises

Solve and check:

1. $\dfrac{2}{x+3}=\dfrac{1}{3-x}$.

2. $\dfrac{6}{y-1}-\dfrac{3}{y+1}=\dfrac{6}{1-y^2}$.

3. $\dfrac{2x+1}{2x-1}+\dfrac{8}{1-4x^2}=\dfrac{2x-1}{2x+1}$.

4. $\dfrac{3x}{2x-5}+\dfrac{2x}{5-2x}=\dfrac{x-1}{2x+5}$.

5. $\dfrac{x+3}{x-1}+\dfrac{7x-4}{1-x^2}=\dfrac{x-1}{x+1}$.

6. $\dfrac{6}{n+2}+\dfrac{n^2}{n^2-4}=-\dfrac{n+1}{2-n}$.

Selection or Multiple Choice Test

Select the correct root for each of the following equations:

	Equations	Possible Roots
1.	$3x + 2 = 4x - 1.$	$1, 2, 3, -2, -1, 4.$
2.	$x + 1 = 3 - (x + 6).$	$-1, -3, -2, 1, 3, 4, 2.$
3.	$\frac{x}{2} = \frac{x}{3} + \frac{1}{2}.$	$\frac{2}{3}, 3, \frac{3}{2}, 1, \frac{2}{3}.$
4.	$\frac{x-1}{4} = \frac{1}{2} - x.$	$\frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}, 3.$
5.	$\frac{x}{x-2} = \frac{1}{3}.$	$-1, -2, 1, 3, \frac{1}{2}.$
6.	$\frac{4}{x-1} = \frac{8}{x+1}.$	$3, -3, 4, -4, 2, \frac{1}{3}.$
7.	$\frac{6}{x-2} = \frac{24}{3x-3}$	$5, -5, \frac{2}{5}, -\frac{2}{5}, \frac{5}{2}, -\frac{5}{2}.$
8.	$\frac{6x-9}{3} = \frac{2x}{4}.$	$\frac{1}{3}, -\frac{1}{3}, 3, -3, 2, -2.$
9.	$\frac{y}{6} = \frac{1}{6} - y.$	$\frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{3}, -\frac{1}{5}.$
10.	$\frac{3}{n+1} - \frac{5}{n-1} = \frac{8}{n^2-1}.$	$8, -8, 3, -3, 5.$

100. Decimal Equations. — Fractional equations whose denominators are 10, 100, 1000, etc., such as $\frac{3}{10}x + \frac{8}{10}x = \frac{77}{100}$, are usually written as decimals, thus:

$$.3x + .8x = .77.$$

Multiplying by 100,

$$30x + 80x = 77.$$

Adding, $110x = 77.$

Dividing by 110, $x = .7$

Check: $x = .7$.

$$.3(.7) + .8(.7) = .77$$
$$.21 + .56 = .77$$
$$.77 = .77.$$

Exercises

Solve the following decimal equations. Check.

1. $.6\,x + .8\,x = .56$.

2. $1.8\,x - .4\,x = .42$.

3. $.7\,x - .3\,x = 3.2$.

4. $9\,x - 3.6\,x = 16.2$.

5. $3.5\,y - 9.8 = 2.1\,y$.

6. $3.4\,x + 10.2\,x = 81.6$.

7. $5.6\,x - 6.6 = 2.3\,x - 6.6$.

8. $6.3\,x + 220 = 4.1\,x$.

9. $3.2\,x - 10 = .8\,x - .4$.

10. $8.6\,n + 11.7 = 6.3\,n - 2.1$.

11. $8.4\,n + 6.1 - 2.1\,n = 4.2\,n - 4.4$.

12. $.3\,x - 4.6 = .9 - .25\,x$.

Solve to the nearest hundredth:

13. $.3\,x - .02\,x = 6.4$.

14. $.4\,x - .03\,x = 4.3$.

15. $1.53\,x + .23\,x = 1.804$.

16. $2.31\,x - 1.87\,x = .39556$.

17. $\dfrac{2.3\,x - 2.7}{9} = \dfrac{4.6\,x - 3.6}{7}$.

101. Literal Equations Containing Fractions. — In literal equations, as we learned in section 66, some or all of the known quantities are represented by letters. We solve such equations for the desired letter in the same manner as we solve numerical equations.

Illustrative example I:

Solve $ay = bx$ for y.

Solution:

$$ay = bx.$$
$$y = \frac{bx}{a}. \quad \text{By division.}$$

Illustrative example II:

Solve $\frac{a}{x} - \frac{b}{x} = 5$ for x.

Solution:

$$a - b = 5x. \quad \text{Multiplying by } x.$$
$$5x = a - b. \quad \text{Addition axiom.}$$
$$x = \frac{a-b}{5}. \quad \text{Division axiom.}$$

Check:

$$x = \frac{a-b}{5}$$
$$\frac{a}{x} - \frac{b}{x} = 5$$
$$\frac{a}{\frac{a-b}{5}} - \frac{b}{\frac{a-b}{5}} = 5$$
$$\frac{5a}{a-b} - \frac{5b}{a-b} = 5$$
$$\frac{5a - 5b}{a-b} = 5x$$
$$\frac{5(a-b)}{a-b} = 5$$
$$5 = 5.$$

Oral Exercises

Solve for x or y in each of the following. Check each root.

1. $mx = cm$.
2. $ax = ab$.
3. $cx = bc$.
4. $abcx = abcd$.
5. $2\,abx = 4\,abc$.
6. $6\,bcx = 24\,bcd$.
7. $9\,Kx = 7.2\,K$
8. $\frac{x}{3} = a$.
9. $\frac{x}{h} = k$.
10. $\frac{b}{x} = c$.
11. $\frac{d}{x} = m$.

12. $cx + 1 = 0$.
13. $b - dx = c$.
14. $0 = dx - b$.
15. $2x + b = c$.
16. $5x - 6k = 26k$.
17. $x(a + b) = a^2 - b^2$.
18. $ax + bx = a + b$.
19. $\frac{X + 3a}{5} = 6a$.
20. $Hy + Ky = 5(H + K)$.
21. $x(a + b + c) = 3(a + b + c)$.

Written Exercise

Solve each of the following for x or y. Check as directed by your teacher.

1. $6x = b$.
2. $bx = c$.
3. $-7x = b$.
4. $-9x = c$.
5. $ax = a^2$.
6. $6ax = 54a^3$.
7. $\frac{x}{b} = 2$.
8. $y - 3 = b$.
9. $6 + x = a$.
10. $\frac{x}{a} = 5$.
11. $b = 6x$.
12. $a = by$.
13. $1 = ay$.
14. $3a - y = 0$.
15. $0 = y - 3c$.
16. $by - 1 = 0$.
17. $a - cy = 0$.
18. $a = by$.
19. $by = \frac{1}{c}$.
20. $\frac{e}{x} = n$.
21. $b = \frac{ay}{m}$.
22. $6x + 5a = 6a$.
23. $-5x + c = d$.
24. $5x + 3b = x + 9b$.
25. $7bx - 2b = 3bx + 6b$.
26. $9by - a = by - 9a$.
27. $2x - 7b = 6a$.
28. $2bx + b = 4bx - 3b$.
29. $2(x - 3a) = 12a$.
30. $6a = 3a - (y - 2a)$.
31. $6(a - y) = 7(a - 2y)$.
32. $5(by - c) = 6b(y - c)$.
33. $4ax = c - (6d + c - ax)$.
34. $a(y + m) = a^2 + 3am$.
35. $a(y - a) = -b(y + b)$.
36. $ay - 3(y - a) = 4a - 3$.
37. $7y + 4(y - 3b) = 9(y + 2b)$.

Optional

38. $bx - cx = (b - c)^2$.

39. $\frac{2y}{a} = \frac{6d}{3a}$.

40. $\frac{y}{b} = \frac{a}{3b}$.

41. $\frac{by}{c} = \frac{h}{k}$.

42. $\frac{b}{y - b} = 3$.

43. $7 = \frac{9b}{x - b}$.

44. $\frac{y}{2} - \frac{b}{2} = \frac{b}{4}$.

45. $\frac{1}{ab} + 1 = \frac{1}{x} + \frac{ab}{x}$.

46. $\frac{x}{h} + \frac{x}{k} = h + k$.

47. $x(a - b) = a^2 - b^2$.

48. $Hx + K^2 = H^2 - Kx$.

49. $b(b - x) = c(c - x)$.

50. $\frac{d^2}{x} - d = \frac{a^2}{x} + a$.

51. $\frac{c}{x} + \frac{5c}{x} = \frac{7}{8}$.

52. $\frac{a}{x} - 2 = \frac{b}{x} - 5$.

53. $\frac{hk}{x} = km + n + \frac{1}{x}$.

54. $\frac{m+n}{x} - a = \frac{m-n}{x} + c$.

55. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1$.

56. $\frac{x + 1}{x - 1} = \frac{m + n}{m - n}$.

57. $\frac{2b + x}{2b - x} = \frac{5b + x}{5b - x}$.

102. Formulas as Fractional Equations. — As we said in section 67, the formula is the most important type of literal equation. Formulas containing fractions are common in algebra, geometry, and physics. They sometimes make trouble, because pupils think of them as something complicated. But they are solved just as if the unfamiliar π's, V's, C's, S's, l's, and r's were the common a's, b's, x's, and y's. Below are some of the commonest fractional formulas.

1. $C = \frac{5}{9}(F - 32)$. For changing temperature from Fahrenheit to Centigrade scale.

2.	$S = \frac{1}{2}at^2$.	For finding distance, when time and acceleration are given.
3.	$S = \frac{n}{2}(a + l)$.	Used in series, a later topic of algebra.
4.	$S = \frac{rl - a}{r - 1}$.	Used in series, later in algebra.
5.	$\frac{w_1}{w_2} = \frac{l_2}{l_1}$.	Connecting weights on a balance with length of arms.
6.	$V = \frac{1}{3}\pi r^2 h$. $\pi = \frac{22}{7}$.	For finding the volume of a cone.
7.	$V = \frac{4}{3}\pi r^3$.	For finding volume of sphere.
8.	$V = \frac{1}{6}\pi d^3$.	For finding volume of sphere.
9.	$\frac{V_1}{V_2} = \frac{P_2}{P_1}$.	Connecting the volume of gas with its pressure.
10.	$A = \frac{(b + b')h}{2}$.	For finding the area of a trapezoid.
11.	$C = \frac{en}{R + nr}$.	Used in electrical measurements.
12.	$A = \frac{\pi r^2 E}{180}$.	For finding the area of a triangle on a sphere.

Illustrative example:

The formula $A = \frac{1}{2}bh$ expresses the area (A) of a triangle in terms of its base (b) and its altitude (h). If, however, you are asked to find the base of a triangle when the area and height are given, you should first *solve the formula for* b, that is, you should transform the equation so that b stands alone as the left member.

Thus $A = \frac{1}{2}bh$

$2A = bh$. Multiplying by 2.

Or $bh = 2A$

$$b = \frac{2A}{h}.$$

In a formula any quantity for which you are asked to solve the formula is treated as the unknown quantity.

Suppose that you are to find b, when $A = 24$ and $h = 8$.

$$b = \frac{2A}{h}$$

$$= \tfrac{48}{8} \quad \text{By substituting given values.}$$

$$= 6.$$

Exercises

In the following state how the changes in the right member affect the left member. See page 266.

1. (a) Solve formula 1 for the value of F.
(b) Compute the value of F when $C = 100°$.
(c) $C = 0°$.
(d) $C = -17\frac{7}{9}°$.

2. (a) Solve formula 2 for the value of a.
(b) Find the value of S if $a = 32$ and $t = 4$ seconds.
(c) if $a = 32$ and $t = 8$ seconds.
(d) if $a = 32$ and $t = 16$ seconds.

3. (a) Solve formula 3 for the value of n; for a; for l.
(b) Find the value of S if $n = 10$, $a = 3$, and $l = 30$.
(c) if $n = 20$, $a = 3$, and $l = 30$.
(d) if $n = 30$, $a = 3$, and $l = 30$.

4. (a) Solve formula 4 for r; for l; for a.
(b) Find S if $a = 4$, $r = 2$, and $l = 128$.
(c) if $a = 4$, $r = 2$, and $l = 256$.

5. (a) Solve formula 5 for l_1.
(b) Find l_1 if $W_1 = 10$ lb., $W_2 = 24$ lb., $l_2 = 4$ in.
(c) $W_1 = 20$ lb., $W_2 = 24$ lb., $l_2 = 4$ in.
(d) $W_1 = 20$ lb., $W_2 = 48$ lb., $l_2 = 8$ in.

6. (a) Solve formula 6 for h.

(b) Find V to the *nearest* hundredth if $h = 2.36$, $r = 1.15$.

(c) $h = 2.36$, $r = 2.3$.

(d) $h = 4.72$, $r = 2.3$.

7. In formula 7 compute V to the *nearest* hundredth if $r = 2.08$; if $r = 4.16$; if $r = 6.24$.

8. In formula 8 compute V to the *nearest* tenth if $d = 5.44$; if $d = 10.88$; if $d = 16.32$.

9. (a) Solve formula 9 for P_2; for V_2.

(b) Compute P_2 if $V_1 = 12$, $V_2 = 60$, $P_1 = 20$.

(c) if $V_1 = 12$, $V_2 = 60$, $P_1 = 40$.

(d) if $V_1 = 24$, $V_2 = 60$, $P_1 = 20$.

10. (a) Solve formula 10 for h.

(b) Compute A if $b = 30$, $b' = 18$, and $h = 12$.

(c) if $b = 30$, $b' = 18$, and $h = 24$.

(d) if $b = 60$, $b' = 18$, and $h = 12$.

11. Solve formula 11 for e; for n; for R.

12. (a) Solve formula 12 for E.

(b) Compute A if $r = 16$ and $E = 36$.

(c) if $r = 16$ and $E = 72$.

(d) if $r = 32$ and $E = 72$.

13. In the formula $C = \frac{5}{9}(F - 32)$, what is the effect on C if F is increased? if F is decreased? If the value of F is known, is the value of C thereby known? May we then say that the value of C depends upon the value of F?

14. Answer a similar set of questions for the formula $S = 16\,t^2$.

15. In like manner discuss the formula $K = \frac{10}{n}$.

16. If *s. p.* (selling price) = *c* (cost) + *g* (gain), solve for the value of *c*; for the value of *g*.

17. If $\frac{g(\text{gain})}{c(\text{cost})} = r(\text{rate \% of gain})$, find the value of *g*; of *c*.

18. If *m. p.* $(1 - r\%) = s.\ p.$ *m. p.* = marked price; $r\%$ = discount; *s.p.* = selling price. Solve for *m. p.*

19. If $i = prt$,

where i = interest, r = rate %,
p = principal, t = time in years,

solve this for p; for r; for t.

20. If $V = abc$,

where V = volume of a rectangular solid,
a = height of solid,
b = width of solid,
c = length of solid,

solve for a; for b; for c.

21. In division, the dividend equals the quotient multiplied by the divisor and the result added to the remainder. Write this as a formula.

22. Write a similar formula for multiplication; for the proof of subtraction.

Matching Formulas

Copy the left column and match the proper formula of the right-hand column with its corresponding statement of the left column:

1. The formula for area of a rectangle	$s = c + g$
2. The formula for the area of a parallelogram	$S = 6e^2$
3. The area of a circle is	$V = lwh.$

4. The circumference of a circle is	$A = \frac{a}{2}(b + b')$
5. The simple interest on a sum of money is	$A = bh$
6. The area of the surface of a cube	$h = \frac{2A}{b}$
7. The volume of a cube	$A = s^2$
8. The volume of a rectangular solid	$d = rt$
9. The area of a trapezoid	$A = lw$
10. The area of a square	$V = \frac{4}{3}\pi r^3$
11 The distance one travels	$V = e^3$
12. The volume of a sphere	$V = \frac{1}{3}\pi r^2 h$
13. The volume of a cone	$C = 2\pi r$
14. The altitude of a triangle	$A = \pi r^2$
15. The selling price	$I = prt$

Selection or Multiple Choice Exercises

Copy each of the following and place after each the result that you consider to be the correct answer:

1. In the formula $A = lw$, if $l = 6$, $w = 12$, then $A = 48, 60, 72, 24$.

2. In the formula $r = \frac{d}{t}$, if $d = 200$, $t = 5$, then $r = 20, 40, \frac{1}{40}, 10$.

3. In the formula $l = 12\,e$, if $e = 3$, then $l = 36, 4, \frac{1}{4}, 15$.

4. In the formula $A = \frac{bh}{2}$, if $b = 8$, $h = 6$, then $A = 48, 24, 7, \frac{1}{24}$.

5. In the formula $w = \frac{A}{l}$, if $A = 140$, $l = 20$, then $w = 2800, \frac{1}{7}, 7, 16$.

6. In the formula $V = lwh$, if $l = \frac{1}{4}$, $w = 8$, $h = 6$, then $V = 144,\ 12,\ 14\frac{1}{4},\ 3$.

7. The formula $A = \pi r^2$ is the formula for the area of a sphere, a circle, a triangle.

8. In the formula $g = s - c$, if $s = 17¢$, $c = 12¢$, then $g = 5¢,\ 29¢,\ 204¢$.

9. The formula for the perimeter of a rectangle is $p = 2\pi r,\ l + w,\ 2l + w,\ 2l + 2w$.

10. In the formula $V = Bh$, if $B = 80$, $h = \frac{1}{4}$, then $V = 320,\ 80\frac{1}{4},\ 20$.

Oral Exercises

1. $\frac{2}{3}$ of a certain number is 13. What is the number?

2. When John has gained $\frac{1}{4}$ of his present weight, he will weigh 125 pounds. What is his weight now?

3. After I have earned half as much money as I now possess I shall have \$12. How much have I now?

4. Henry lost $\frac{1}{4}$ of his marbles and had 30 left. How many had he at first?

5. I am half as old as my father and the sum of our ages is 72 years. How old is each?

6. 27 is $\frac{2}{7}$ of what number more than the number?

7. A watch cost \$50, which was $\frac{3}{7}$ more than the cost of a clock. What was the cost of the clock?

8. John had \$7.60, which was $\frac{11}{27}$ more than Mary had. How much had Mary?

9. My sales this year are \$57,000, or $\frac{3}{16}$ more than my sales last year. What were my sales last year?

Algebraic Shorthand

State an equation for each of the following:

1. If 3 be added to a certain number, then $\frac{3}{5}$ of the sum will equal 21.

2. $\frac{3}{4}$ of a certain number added to 24 equals 60.

3. The excess of a number over 12 equals $\frac{1}{3}$ of that number.

4. One fourth of a certain number added to four times the number equals 34.

5. One fourth of a certain number exceeds 3 by as much as the number exceeds one eighth of the number.

6. Interpret in words $\frac{x}{5} - \frac{x}{8} = 3$.

Exercises

Problems Leading to Fractional Equations

Number Problems

1. What number is that which increased by $\frac{5}{9}$ of itself equals 28?

2. What number is that which increased by 75% of itself equals 63?

3. One fourth of a number plus one half of the number equals 27. Find the number.

4. Two men start in business, investing $15,000. One man invests two thirds as much as the other. How much will each invest?

5. The sum of the seventh and eighth parts of a certain number is 13 less than half the number. Find the number.

6. One third of a certain number exceeds 15 by as much as the number exceeds 63. What is the number?

7. The sum of two numbers is 44, and five eighths of the larger exceeds the smaller by eight. What are the numbers?

8. Divide 451 into two parts such that if 14 is added to one part and the other part is divided by 14, the results are the same.

9. 450 is separated into 2 parts in such a way that the larger part divided by the smaller part equals 4. What are the two parts?

10. Separate 175 into two parts such that one of them is $\frac{1}{6}$ as large as the other.

11. What number is it whose double is 24 more than its half?

12. What number is it whose half is 4 in excess of its third?

13. What number must be added to the numerator and also to the denominator of the fraction $\frac{12}{29}$ in order that the resulting fraction may equal $\frac{1}{2}$?

14. One fifth of the quantity $(3x + 2)$ divided by x equals the quotient of $(3x - 2)$ divided by $(5x - 4)$. What is the value of x?

15. A baseball team has won 50 games out of the 75 games played. How many of the 25 games it still has to play must it win in order to have a percentage of 60 (or 60%)?

16. The numerator of a fraction is 7 less than the denominator. If 4 is subtracted from the numerator and 1 added to the denominator, the resulting fraction equals $\frac{1}{3}$. Find the fraction.

17. A certain number divided by 8 equals 4 as a quotient with 5 as a remainder. Find that number.

Hint: $\frac{x}{8} = 4 + \frac{5}{8}$.

18. A certain number when divided by 13 gives 17 as a quotient and a remainder of 11. What is the number?

19. What number divided by $\frac{1}{2}$ gives a quotient of 15 and a remainder of $\frac{1}{3}$?

20. What number must be added to each of the terms of the fractions $\frac{3}{4}$ and $\frac{31}{36}$ that the results shall be equal?

21. The dividend plus the remainder 5 is $4x^2 - 31x + 26$. The divisor is $4x - 3$. Find the quotient.

Averages

1. A merchant pays d dollars for n yards of velvet. What is the price per yard?

2. If an automobile goes m miles in 10 hours, what is the average rate per hour?

3. If s dollars represents the total sales of 5 salesmen, what is the average sales per salesman?

4. What is the average weight of three men whose weights are x, y, and z pounds?

5. What is the average yearly saving of three boys whose yearly savings are d, e, and f dollars?

6. Mary received marks of A, A, B, and C in her four subjects. What was her average?

Problems

1. A boy passes one subject with a mark of 86. What mark must he attain in another subject in order that his average shall be 90?

Hint: $\dfrac{86 + x}{2} = 90.$

2. A girl passes three subjects with marks of 76, 80, and 72. What mark must she attain in a fourth subject in order that her average shall be 80?

3. A boy has a mark of 90 in algebra, 86 in Latin, and 80 in French. What must he get in English to have an average of 85?

Loading Coal

4. A coal dealer brings me 4 loads of coal weighing 2500, 2300, 2600, and 2200 pounds. How much must he bring me in the fifth load to make 7 tons in all?

Geometry Problems

1. A post is a fifth of its length in the ground, one half of its length in water, and 9 feet above water. How long is the post?

2. The width of a rectangle is $\frac{3}{4}$ of its length. If we increase the length by 8 feet and the width by 6 feet, the area is increased by 96 square feet. What were the dimensions of the original rectangle?

3. The width of a rectangle is $\frac{5}{8}$ of its length. If the width were decreased by 2 feet and the length increased by 4 feet, the area would be increased by 8 square feet. Find the dimensions of the rectangle.

4. The length of a rectangular field exceeds its width by 50%. After a strip 3 rods wide had been plowed around the field there remained 72 sq. rods to plow. Find the length and width of the field.

Algebraic Representation

1. If tennis balls cost r cents a dozen last year and the price has advanced 60¢ a dozen, how much will a half dozen cost at the present rate?

2. If a man has x dollars and spends $\frac{1}{5}$ of that sum, how many dollars has he left?

3. If one acre of potatoes is worth d dollars, what is $\frac{1}{3}$ of an acre worth? $\frac{3}{5}$ of an acre?

4. State as a fraction 25% of y; 50% of y; $33\frac{1}{3}$% of y; 125% of y; 1% of y.

5. What is the annual income on d dollars at 6%? On $\frac{3}{4}$ of d dollars at 7%?

Business Problems

1. A used car dealer sold an automobile for $300, thereby making 25% of its cost. Find the cost.

2. A man invests a part of $1800 at 5% and the rest at 6%. His total income from both investments is $100. How much had he invested at each rate?

Hint: Let x = amount invested at 5%.

Then $.05\,x + .06\,(1800 - x) = 100$.

3. A sum of money invested at 4% interest produces $300 less interest than it would if invested at 6%. Find the amount invested.

4. A man's investments produced \$1060 annually. If $\frac{1}{2}$ of his property is invested at 5%, $\frac{2}{5}$ of his money at 6%, and the balance at 4%, what are his savings?

5. A man invests $\frac{1}{3}$ of his money at 5% interest, $\frac{1}{4}$ of his money at 6% interest, and the balance of his money at 4% interest. If his annual income is \$580, how much money has he invested at each rate?

6. A boy buys marbles at the rate of 5 for 2 cents. He sells half of them 2 for a cent and the other half at $\frac{1}{3}$ of a cent each and gains 10 cents. How many marbles had he?

Coin Problems

1. If 27 coins composed of nickels and quarters have a total value of \$4.95, how many are there of each?

2. Mr. Burrows paid a debt of \$5.85 with 33 coins, using only dimes and quarters. How many coins of each kind did he use?

3. When Edgar counted the money in his bank, he found that he had \$2.40 in nickels, dimes, and quarters. There were three more dimes than quarters, and twice as many nickels as dimes. How many coins of each kind did he have?

Work Problems

1. If a boy can mow a large lawn in 3 hours, what part of it can he mow in 1 hour? In x hours?

2. Copy and complete

Time to Do a Job	Part of the Work That Can Be Done in 1 Hour
24 hours	
6 hours	
h hours	
$(h + 2)$ hours	
$(x - 1)$ hours	

3. John can do a piece of work in 5 days and Henry in 8 days. In how many days can they do the work if they work together?

Hint: The quantity that we know the least about is the number of days it will take them when both are working together.

Let x = the number of days, both working together.

	Time Alone	Amount Done in 1 Day
John . . .	5	$\frac{1}{5}$
Henry . .	8	$\frac{1}{8}$
Together .	x	$\frac{1}{x}$

John can do $\frac{1}{5}$ of the work in 1 day and Henry $\frac{1}{8}$ of the work in 1 day. If they require x days when working together, then $\frac{1}{5} + \frac{1}{8} = \frac{1}{x}$, etc.

4. Mr. Jones can plow a field in 7 days, but Mr. Brown can do the same work in 4 days. How many days will be required if both men work at the job?

5. Frank can do in 8 days a task which he and James together can do in 3 days. How many days would be required for James to do the work alone?

6. A business man found that one of his girls could type a certain number of form letters in 5 hours, while another girl required $\frac{4}{5}$ as long. How long would it take them both working together?

7. A printing press did a certain piece of work in 4 hours; when a second press was installed, a similar job was completed by both machines in $2\frac{1}{2}$ hours. In how many hours could the second machine alone have done the work?

A Printing Press

8. One pipe stream can fill a tank in 5 hours alone. Another pipe stream can fill the tank in 8 hours alone. In how many hours will the two pipe streams together fill the tank?

9. A tank can be filled by one pipe in 2 hours, by another in 3 hours, and by another in 4 hours. How many hours will it take the three pipes, all open together, to fill the tank?

10. A tank can be filled by one pipe stream in 3 hours, and by another pipe stream in 4 hours. It can be emptied by another pipe stream in 5 hours. How many hours will it require to fill the tank if all pipes are open?

Hint: $\frac{1}{3}+\frac{1}{4}-\frac{1}{5}=\frac{1}{x}$, etc.

Motion Problems

Algebraic Representation

1. In how many hours can a boy walk m miles at the rate of c miles per hour?

2. If a man's rate of rowing in still water is S miles an hour and the current flows at the rate of C miles an hour, express the man's rate (1) when rowing with the current, (2) when rowing against the current.

An Airplane

3. An airplane can go 100 miles an hour in still air. How fast can it go ahead against a wind blowing x miles an hour? With the wind?

4. If an automobile goes m miles per hour, what is the rate in feet per minute?

5. One automobile goes x miles an hour. Another travels 10 miles an hour faster. How fast does it travel?

6. Two trains leave a town at the same time and travel in the same direction; the faster one runs m miles per hour and the slower runs s miles per hour. In how many hours will they be k miles apart?

7. If a freight train travels $\frac{2}{3}$ as fast as an automobile, and r stands for the rate of the automobile

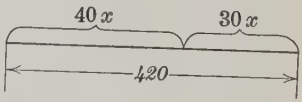
(a) What represents the rate of the freight train?

(b) How many hours will it take each to go 100 miles?

Time, Rate, and Distance Problems

1. Two automobilists start out from towns 420 miles apart and travel toward each other at the rate of 30 and 40 miles an hour respectively. In how many hours will they meet?

Solution:



Let x = number of hours each travels.

$$d = rt.$$

	Rate	Time	Distance
First Autoist . .	30 m. per hr.	x	$30\,x$
Second Autoist .	40 m. per hr.	x	$40\,x$

Then $40\,x + 30\,x = 420$. Solve.

2. Two automobiles travel in opposite directions from the same point, at rates of 25 miles and 30 miles an hour. In how many hours will they be 275 miles apart?

3. Two men walk 3 and 4 miles per hour respectively. They start from two points 35 miles apart and walk toward each other. In how many hours will they meet?

4. An airplane travels twice as fast as an autoist. The former requires 1 hour longer to go 400 miles than the latter takes to go 160 miles. What is the rate of each?

Solution:

Let x = rate in miles per hour of autoist.

Then $2x$ = rate in miles per hour of airplane.

$$d = rt.$$

	Rate	Time	Distance
Automobile. .	x	$\frac{160}{x}$	160 miles
Airplane . .	$2x$	$\frac{400}{2x}$	400 miles

The problem states that the time required by the airplane is 1 hour less than that required by the automobile. This gives us the equation

$$\frac{400}{2x} - \frac{160}{x} = 1. \quad \text{Complete.}$$

5. An automobile travels six times as fast as a bicycle. The former goes 168 miles in 2 hours less time than the latter requires to go 42 miles. What is the rate of each?

6. Two automobilists start from two points 330 miles apart at the same time and travel toward each other. The first goes 6 miles an hour faster than the second. If they meet in 5 hours, what is the rate of each?

Hint:

Let x = rate in miles per hour of first.

Then $x + 6$ = rate in miles per hour of second.

	Rate	Time	Distance
First	x	5	$5x$
Second . . .	$x + 6$	5	$5(x + 6)$

7. One man starts from Chicago in an automobile and travels toward New York, 1000 miles distant, at an average rate of 24 miles an hour. Two hours later a second man starts from Chicago and goes toward New York at an average rate of 40 miles an hour. In how many hours will the latter catch the former?

Hint:

Let x = number of hours latter travels.

$x + 2$ = number of hours former travels.

	Rate	Time	Distance
First	24	$x + 2$	$24(x + 2)$
Second . . .	40	x	$40\,x$

Distances are the same when they meet. Complete.

8. In going a certain distance a train which runs 48 miles an hour requires 2 hours longer than a train which runs 60 miles in an hour. What is the distance?

9. An automobilist traveled to a certain point and returned in 12 hours. If the total distance was 240 miles, and the rate returning was 30 miles per hour, at what rate did he go?

Hint: Let x = rate in miles per hour in going.

10. A man can walk 4 miles an hour. How far may he walk into the country and ride back on a trolley car, which runs 20 miles an hour, if he must be home in 3 hours?

11. The rate of a passenger train is $2\frac{1}{2}$ times that of a freight train. It left a certain station 3 hours after the freight train and overtook it after traveling 100 miles. What was the rate of the freight train?

12. Two boys have a motor boat which will travel 20 miles an hour in still water. How far may they go up a stream flowing 5 miles an hour and be back in 3 hours?

Answer to tenths.

13. How far down the same stream may the boys go if the round trip requires 4 hours?

Mixture Problems

Oral

1. If 12 gallons of water and 3 gallons of glycerine are mixed, what part of the mixture is glycerine?

2. If an alloy is composed of 4 parts of gold and 3 of silver, what part of the whole is the silver?

3. If an alloy of copper contains 7 parts of tin and 3 parts of copper, what per cent of the whole is the tin?

Written

4. A coffee firm wishes to mix two grades of coffee of the 35 cent and 60 cent kinds in order to obtain a mixture of 50 pounds which they may sell at 45 cents a pound. How many pounds of each will they use?

Solution:

Let x = number of pounds of 35-cent coffee.

Then $50 - x$ = number of pounds of 60-cent coffee.

Hence $35\,x + 60(50 - x) = 50 \cdot 45$. Complete.

5. A candy firm wishes to mix some 40-cent candy with some 80-cent candy so as to have a mixture of 100 pounds that they may sell at 50 cents a pound. How many pounds of each kind of candy shall they use?

6. In a mixture of 10 quarts, of which 3 quarts are alcohol, I wish to add enough water so that the resulting mix-

ture shall contain 20% alcohol (or in other words shall be a 20% solution). How many quarts of water should be added?

Hint: Let x = the number of quarts of water to be added.

Then $x + 10$ = the number of quarts in the resulting mixture.

$\frac{1}{5}$ = the part of the final mixture which shall be alcohol.

3 = number of quarts of alcohol in that mixture.

Hence, $3 = \frac{1}{5}(x + 10)$.

7. Of a mixture of 20 quarts 80% is alcohol. How many quarts of water must be added to produce a mixture 70% pure?

8. Of a mixture containing 7 quarts, 5 quarts are alcohol. How many quarts of water must be added to produce a mixture containing 25% alcohol?

9. A man finds that in the radiator of his car there are six quarts of a 15% solution of alcohol. How much more alcohol must be added to make it a 28% solution?

10. If water is to be added to 40 quarts of milk containing 4% butter fat so as to produce a mixture containing 3% butter fat, how many quarts of water must be added?

11. A chemist has 10 quarts of a 10% solution of acid. He wishes to add enough water to it to produce a 4% solution. How many quarts of water must he add?

12. If to 8 quarts of milk containing 4% butter fat, a certain number of quarts of cream containing 20% butter fat is to be added so as to produce a mixture containing 16% butter fat, how many quarts of the cream must be added?

13. Gold, when weighed in water, loses $\frac{1}{19}$ of its weight, and silver, when weighed in water, loses $\frac{2}{21}$ of its weight. If a mixture of gold and silver weighing 40 pounds in air loses

3 pounds when weighed in water, how many pounds of gold and silver are there in the mixture?

14. A chemist has 5 quarts of an 8% solution of acetic acid. How many quarts of pure acid must he add to raise it to a 25% solution?

15. If 20 pounds of sea water contain 1.6 pounds of salt, how many pounds of pure water must be added to produce a mixture containing 5% salt?

16. A druggist has a 6% solution and also a 15% solution of a certain medicine. How much of the stronger solution must he add to a liter of the weaker to obtain a 10% solution?

A Playground See-Saw

Lever Problems

You are familiar with the see-saw (teeter board) at the playground. This illustrates the lever, used in physics.

Law of the lever:

The product of the weight W by the length of its arm (WF) = product of the weight w and its arm (wF).

Illustrative example:

If John, who weighs 40 pounds, and sits at w, 8 feet from the fulcrum, just balances his sister Mary, who sits at W and weighs 80 pounds, how long is WF?

Solution:

Let x = the length of WF in feet.
Then $80\,x = 8 \cdot 40$. Complete.

Exercises

1. If Henry weighs 50 pounds and is 10 feet from F, how much does his sister weigh if she just balances him and sits 4 feet from F?

2. A man wishes to lift a 300-pound stone with a crowbar 6 feet long over the fulcrum F. The bar is placed so that the stone is 1 foot from the fulcrum. How much force downward must the man use to raise the stone?

3. What weight is required on the bar in example 2 to raise a stone weighing 500 pounds and placed $\frac{1}{2}$ foot from F?

4. Two boys, weighing 90 pounds and 150 pounds respectively, use a plank 16 feet long for a see-saw. How far from the heavier boy must F be?

Hint: Let x = distance from heavy boy.
Then $16 - x$ = distance from lighter boy.

NOTE TO TEACHER. If it is desired to take up ratio, proportion, and similar triangles at this point, a treatment of these topics will be found on pages 409–430.

Miscellaneous Problems

1. A man willed half his farm to his wife, one fourth to his son, one fifth to his daughter, and the balance, which was 15 acres, to his cousin. How many acres in the farm?

2. What sum of money increased by $\frac{1}{4}$ of itself and by \$25 equals \$200.50?

3. A 15-foot length is to be divided into two parts such that one part shall be $\frac{2}{3}$ of the length of the other part. What shall be the length of each part?

4. A man spent $\frac{1}{3}$ of his money and 6 dollars more, and found that he had 8 dollars more than half of it left. How many dollars had he at first?

5. A man lost $\frac{2}{3}$ of his fortune and \$500, and found that he had $\frac{1}{4}$ of his original fortune left. How much money had he at first?

6. John is now half as old as his father; twelve years ago he was one third as old. How old is each?

7. Harry is $\frac{2}{3}$ as old as his sister. Twenty years hence he will be $\frac{7}{8}$ as old. How old is each?

8. A sum of \$1500 was to be divided equally among a certain number of persons. If there had been 5 persons more, each would have received only $\frac{2}{3}$ as much. How many persons were there?

9. The circumference of a rear wheel of a certain wagon is 3 feet longer than the circumference of the front wheel. The rear wheel makes 100 less revolutions than the front wheel in traveling a distance of 6000 feet. Find the circumference of each wheel.

Review

1. Multiply:

$$\frac{x^2 - 6x + 9}{x^2 - 4x + 4} \cdot \frac{(x-2)^3}{x^2 + 8x + 16} \cdot \frac{x+4}{x-3}.$$

2. Reduce to a fraction having one denominator:

$$x^2 + x - 1 - \frac{x+1}{x-3}.$$

3. Add:

$$\frac{x^2 + x + 3}{x+2} + \frac{x^2 - x - 4}{x^2 - 4} - \frac{x^2 - x + 6}{2 - x}.$$

4. Factor: (**a**) $15x^3 - 16x^2 - 15x$.

(**b**) $m^2n^2 + 9m^2n + 18m^2$.

5. Find a number such that its half, plus its third, plus its fourth equals thirty-nine.

6. Show that $\frac{3x^2 + 5x - 16}{x+4}$ does or does not equal $3x - 7 + \frac{12}{x+4}$.

7. What is the Lowest Common Multiple of $x^2 - 4$ and $x^2 - 5x + 6$?

8. If the area of a rectangle is $a^2 - 9b^2$ and its length is $a + 3b$, what represents its width?

9. $A = \frac{1}{2}bh$. Construct a graph of this formula when $h = 10$.

From the graph read the values of A when $b = 1, 2, 4, 6$.

10. Combine $\frac{3}{a-1} + \frac{2}{1-a}$.

11. Is the following statement true or false? (x not 0).

$$(a - 2x)^2 = a^2 + 4x^2$$

If it is false, give the correct value of $(a - 2x)^2$.

12. Solve for x: $\frac{x}{a} + \frac{1}{a} = ax$.

13. Factor $2x^2 + x - 6$.

14. If oranges sell for k cents a dozen, how many dimes should I pay for 36 oranges?

ORANGES

15. 6 more than $\frac{1}{3}$ of a certain number is 2 less than $\frac{1}{2}$ of it. Find the number.

16. Simplify: $\left(\frac{a}{x} - \frac{x}{a}\right) \cdot \frac{2ax}{a + x}$.

17. $a + a + a = ?$ $\quad a \cdot a \cdot a = ?$

18. In the formula $d = 30t$, show that the value of d is doubled if the value of t is doubled. (Use $t = 1, 2$, etc.)

19. If the length of a rectangle is $a + b$ and its width is a, what is its area?

20. If one side of a square is s feet, what does $12s$ represent? Write a formula for the perimeter of this square in inches.

21. $a(x + y + z)$ is a monomial, binomial, trinomial, polynomial. Select the correct word.

22. Divide $a^3 + 8\,b^3$ by $a + 2\,b$ and check by multiplication.

23. If a and b are positive numbers, which is greater, $\frac{a}{b}$ or $\frac{a}{b+1}$?

24. If the numerator of a fraction remains the same, how is the fraction changed if the denominator is decreased?

25. Remove parentheses and collect terms:

$$3\,a - (a - b) + (2\,a - b).$$

26. Simplify: $a^2 - b^2 - 2(a + b)(a - 2\,b)$.

27. Solve the formula $n = \frac{2\,S}{a + l}$ for S.

28. Mary's examination mark in Latin was 78, in English 86, in Science 82, and she wishes to get a grade in algebra high enough so that her average for the four subjects may be 85. What must she get in algebra?

29. Indicate the quotient of the sum of two numbers divided by the difference of their squares and reduce the fraction to lowest terms.

30. If $x^2 - 6\,x + k$ can be divided evenly by $x - 2$, what must be the value of k?

CHAPTER X

GRAPHS OF SIMPLE LINEAR EQUATIONS. SYSTEMS

We have studied simple equations with one unknown. We are ready to proceed to our study of equations having two unknowns. However, before we go on with this study we must learn to picture simple equations in graphs, just as in section 11 we made graphs of formulas. As fractions occur so often in these equations, we do not draw their graphs until we have finished fractional equations.

103. Graphs of Simple Equations. — The main thing in all algebraic graphs is the principle of dependence, which we studied in Chapters I and II. The condition that makes treatment by the algebraic graph possible is that there shall be two changing quantities (variables), one of which is a function of (depends upon) the other. This we have already seen in the formula graph, section 11.

Let us study these graphs through

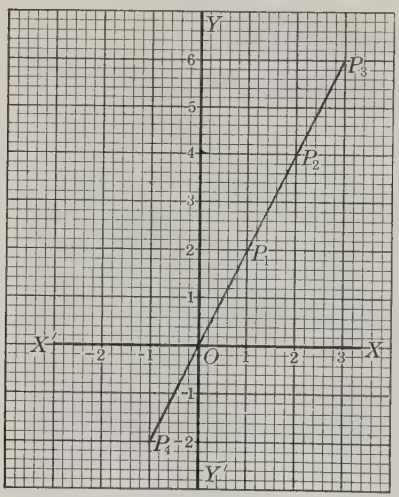
Illustrative examples:

I. Let us draw the graph of the equation $y = 2x$. Here you see *the value of y depends on the value of x*, hence we have the condition which makes an algebraic graph possible.

As both x and y may have negative as well as positive values, we should draw the axes so that the origin is near the center of the graph sheet. The horizontal axis should be lettered XX' and the vertical axis YY', as indicated. Their intersection should be marked O.

In locating points for the graph, values of x are laid off on the horizontal or X-axis, and values of y on the vertical

or Y-axis. *Positive values* of x are laid off to the *right* and *negative* values to the *left* of O. *Positive values* of y are laid off *above* and *negative values below* O.



Since $y = 2x$ always, the points must be so located that the y value of each one is twice its x value. For example, at P_1, $x = 1$ and $y = 2$; at P_2, $x = 2$ and $y = 4$; at P_3, $x = 3$ and $y = 6$; at P_4, $x = -1$ and $y = -2$.

Notice that the graph of $y = 2x$ is a *straight line.*

We shall now graph a more difficult equation.

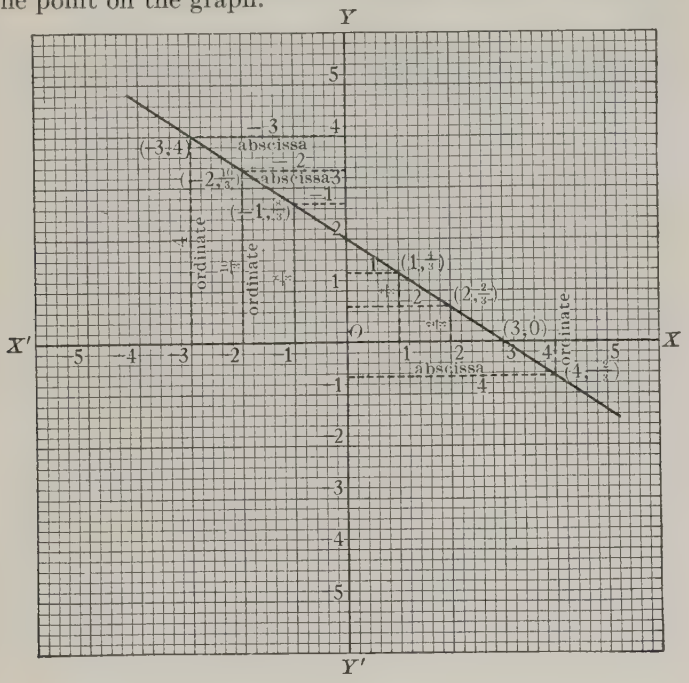
II. Graph the equation $2x + 3y = 6$. When we solve for y, we obtain $$y = \frac{6 - 2x}{3}.$$

Now, if x is given a succession of values, say 0, 1, 2, 3, 4, by substituting these values in the equation $y = \frac{6 - 2x}{3}$, we obtain a corresponding set of values for y, namely, 2, $\frac{4}{3}$, $\frac{2}{3}$, 0, $-\frac{2}{3}$, etc. That is, *the value of y depends on the value of x.* We may arrange these values in tabular form as follows:

x	0	1	2	3	4	-1	-2	-3
y	2	$\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	4

Values of x are laid off on the horizontal axis, called the *x-axis, positive* values to the *right* and *negative* to the *left.*

In like manner values of y are laid off on the vertical or *y-axis*, *positive* values *above* the origin and *negative* values *below*. Each pair of values of x and y enables us to locate one point on the graph.



GRAPH OF EQUATION: $2x + 3y = 6$

To locate the point where $x = 0$, $y = 2$, written (0, 2) we locate a point whose distance to the *right* is 0, and whose distance *up* is 2.

The point $(1, \frac{4}{3})$ may be located by measuring off one unit to the *right* and $1\frac{1}{3}$ units *up*.

The point $(2, \frac{2}{3})$ is located by measuring off a distance of 2 units to the *right* and $\frac{2}{3}$ of a unit *up*.

In like manner we continue with the other points until

we come to the point $(-3, 4)$, which may be located by measuring off 3 units to the *left*, and 4 units *up*.

The distances which we have measured off and which are indicated by the dotted lines in the drawing are called the *coördinates* of the points. The horizontal, or x-coördinate, is called the *abscissa*, while the vertical, or y-coördinate, is called the *ordinate*.

If, now, we join these points, we find that our graph is a straight line. It is a general principle that *every simple equation in two unknowns represents a straight line.* For this reason it is called a **linear equation.**

Heretofore we have referred to equations of the first power or degree of x and y as *simple equations.* However, they are not always simple, but they can always be represented by a straight line. For this reason it is more accurate to refer to them as *linear equations.*

Since we now know that we shall obtain a straight line for every such case, we do not need to locate as many points as in the above example. In fact only two points are necessary to determine a straight line and one other to check it. If the abscissa and ordinate, or coördinates, of any point are given, we can locate that point on the chart field. For example, a point whose abscissa is 2 and whose ordinate is -4 is represented as $(2, -4)$.

Locating such a point is usually called *plotting* that point. Plotting two points enables us to graph a linear equation so that we can find any corresponding values of x and y.

Exercises

1. On a piece of squared paper draw an x-axis and a y-axis. Mark off equal units on each. Locate the following points: (*a*) $x = 3$, $y = 2$. (*b*) $-4, 2$. (*c*) $3, 0$. (*d*) $0, 3$. (*e*) $-4, -2$. (*f*) $0, 0$. (*g*) $-5, 0$. (*h*) $-4, -5$.

2. Locate the points (0, 0); (0, 4) and (3, 0). Connect these points. What have you? What is its area?

3. Plot the following points: (2, 1); (8, 1); (2, 5); (8, 5). Connect these points in order. What have you? What is its area?

Draw the graphs of each of the following linear equations:

Hint: It may be found simpler in graph work to use graph paper ruled 10 squares to the inch, if it can be obtained.

4. $x - 2y = 4$.

5. $3x + y = 3$.

6. $x - y = 1$.

7. $2x + y = 0$.

8. $7x - 2y = 10$.

9. $x - 3y = 8$.

10. $x - y = 0$.

11. $x + y = 0$.

12. $3x + 12y = 0$.

What do you notice about the graph of a line when the numerical term of its equation is 0?

13. $4x - 5y = 20$.

14. $3x = 10y$.

15. $9x - y = 6$.

16. $x = 4$.

17. $x = 6$.

18. $2x = 4$.

19. $y = 6$.

20. $y = -4$.

Hint: y may have any value.

21. Where is the line $x = 0$?

22. Write an equation in x and y whose graph is a straight line. Graph it.

All the problems which we have solved before this chapter have contained only one unknown quantity. But there are many problems which we can solve much more simply by the use of equations containing two or more unknowns.

We shall now take up equations containing two unknowns, and learn how to solve problems by the use of such equations.

104. The Indeterminate Equation. — Let us consider the equation $3x + 2y = 4$, an equation of the *first degree* in *two unknown quantities*, x and y.

The equation is said to be *of the first degree* because each term contains only one unknown, and each unknown is of the first power only.

A pair of values which will satisfy the equation is called a *set of roots*. Thus, $x = 2$, $y = -1$ is a set of roots for the equation $3x + 2y = 4$, for, by substituting these values, we obtain $6 - 2 = 4$, which is an identity. Test and determine whether the following sets of values are roots of this equation:

$$x = 1, y = \tfrac{1}{2}; \quad x = 4, y = -4; \quad x = 0, y = 2.$$

You may determine many more sets of values which will satisfy the equation by simply substituting any convenient number for one of the unknowns, say y, and solving the equation for x. Thus, if 5 is substituted for y, the equation becomes $3x + 10 = 4$, and, solving, we obtain $x = -2$. Hence, $x = -2$ and $y = 5$ is another set of values that will satisfy the equation.

By substituting for y the values indicated and obtaining the corresponding values for x, we make a table of values which will satisfy the equation.

x	1	4	0	-2	$1\frac{1}{3}$	$-1\frac{1}{3}$	6	8	-4	-6
y	$\frac{1}{2}$	-4	2	5	0	4	-7	-10	8	11

The number of such values is indefinite, that is to say, as many may be obtained as are desired. Because it is impossible to determine from the equation which particular pair of values is meant, since there is an indefinite number of pairs which satisfy the equation, such an equation is said to be *indeterminate*.

Exercises

Obtain eight sets of values for the unknown quantities which will satisfy each of the following equations, and in each

case arrange the values in tabular form as in the above example:

1. $x - 3y = 6$.
2. $4x + y = 8$.
3. $3x - 2y = 0$.
4. $x - 6y = -2$.
5. $5x - 2y = 12$.

105. Linear Systems. — Let us consider the pair of linear equations

$$4x - y = 5$$

and

$$x + 2y = 8.$$

If we prepare tables of values which will satisfy these equations, as in the above exercise, we might obtain for the equation $4x - y = 5$:

x	$1\frac{1}{4}$	$1\frac{1}{2}$	2	3	1	$\frac{1}{2}$	0	4	-1	-2	
y	0	1	3	7	-1	-3	-5	11	-9	-13	etc.

and for the equation $x + 2y = 8$:

x	0	8	2	6	4	10	-2	-4	-10	
y	4	0	3	1	2	-1	5	6	9	etc.

A comparison of these tables shows that the set of values $x = 2$, $y = 3$ satisfies *both* equations. If it were feasible for us to complete these tables, writing all possible pairs of values, we should find no other set of values which would satisfy both equations. Such a pair of equations is called a *system of equations*, and the set of values which satisfies both equations constitutes a set of roots for the system. The system is then said to be solved.

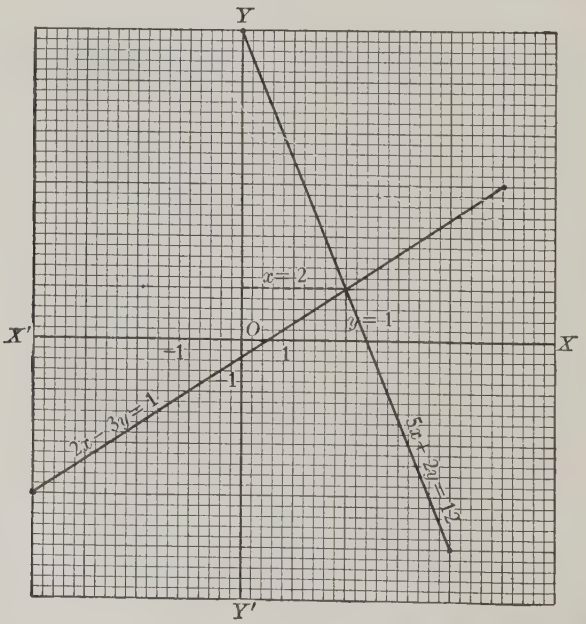
In solving systems of equations like the above it is not necessary to go through the long process just explained in order to obtain the pair of values which will satisfy both equations. We may obtain the roots much more easily by several methods, one of which is the graphic method.

106. Graphic Solution of Linear Systems. — Let it be required to solve by graphic methods the following

Illustrative example:

$$2x - 3y = 1$$
$$5x + 2y = 12.$$

Solution: Plotting at least three points for each equation, we construct the graph of these equations on the same axes.



The coördinates of every point on the first graph are the values of x and y which satisfy the first equation. The coordinates of every point on the second graph are the values of x and y which satisfy the second equation. If the lines intersect, the point of intersection is on both lines; hence, its coördinates satisfy both equations and constitute the solution of the system.

Check by substituting the results obtained in both the given equations.

Exercises

In like manner solve graphically each of the following systems:

1. $5x + y = 4.$
   $x - 2y = 3.$
2. $3x + 4y = -18.$
   $y - x = 6.$
3. $5x - 4y = 8.$
   $2x + y = 11.$
4. $4x + 2y = 0.$
   $x - 5y = -11.$
5. $5x + 2y = 8.$
   $2x - 3y = -12.$
6. $x - y = 3.$
   $2x + y = 12.$
7. $x + 2y = -1.$
   $5x - 4y = 16.$
8. $3x - 5y = 4.$
   $4x + 2y = 1.$
9. $x - 4y = 0.$
   $5x + 7y = 0.$
10. $2x + y = 0.$
    $4x - 2y = 10.$
11. $2x - 3y = 2.$
    $x = 4.$
12. $4x - 5y = 6.$
    $y = 2.$

13. Construct graphs of the three equations $x + y = 5$, $3x - y = 3$, and $x - 4y = -10$ on the same set of axes. Which lines cross and what are the values common to those lines?

14. Construct the graphs of $2x + y = 6$ and $4x + 2y = 6$, and note the relation that exists between the lines. What is your conclusion?

107. Elimination in Systems. — Systems of equations may also be solved algebraically by *eliminating* one of the unknown quantities. The word "eliminate" means "get rid of." This may be accomplished by two or more different methods, the first of which may be called *elimination by addition or subtraction*.

Illustrative example:

Solve $2x + 3y = -4$. (1)
$5x - 2y = 9$. (2)

Solution: $4x + 6y = -8$ (3) By multiplying the 1st equation by 2.
$15x - 6y = 27$ (4) By multiplying the 2d equation by 3.
$19x = 19$ (5) Adding the 3d and 4th equations.
Therefore, $x = 1$. Dividing by 19.

Substitute $x = 1$ in either of the given equations to obtain y. Using the 2nd equation, we have

$$5 - 2y = 9.$$
$$-2y = 4.$$
$$y = -2.$$

Hence, the roots of this equation are $x = 1$, $y = -2$.

Check:
$$2 - 6 = -4.$$
$$5 + 4 = 9.$$

Rule. — (1) *If necessary, multiply either or both of the given equations by numbers which will make the coefficients of one of the unknown quantities alike in both equations.*

(2) *If these like coefficients have like signs, subtract one equation from the other; if they have unlike signs, add the equations.*

(3) *The result will be an equation in one unknown. Solve this equation.*

(4) *Substitute the value thus obtained in one of the given equations to obtain the value of the other unknown.*

(5) *Check both roots in both equations.*

Oral Exercises

Find the value of x and y if:

1. $x + y = 5.$
   $x - y = 1.$
2. $x + y = 6.$
   $x - y = 4.$
3. $x - y = 6.$
   $x + y = 10.$
4. $x + y = 12.$
   $x - y = 2.$
5. $x - y = 10.$
   $x + y = 20.$
6. $x - y = 5.$
   $x + y = 7.$
7. $x - y = 8.$
   $y = 6.$
8. $x + y = 8.$
   $2x = 6.$
9. $2x = 10.$
   $x + y = 8.$
10. $x + y = 12.$
    $3x = 12.$
11. $x + y = 15.$
    $4x = 16.$
12. $x + 3y = 12.$
    $3y = 6.$

Exercises

Solve each of the following systems, and check the solutions in the first 6 examples. The check is valueless unless the values obtained are substituted in *both* equations.

1. $x - 3y = 2.$
   $2x + 3y = 13.$
2. $5x + 3y = 14.$
   $2x + y = 6.$
3. $x - 2y = -13.$
   $3x + y = -4.$
4. $y - 6x = 1.$
   $5x + y = 12.$
5. $2x - y = 9.$
   $5x - 3y = 14.$
6. $y + 4z = 9.$
   $2y - 8z = 6.$
7. $6x + 3y = 4.$
   $10x - 6y = 3.$
8. $4x + y = -4.$
   $10x + 3y = -9.$
9. $8x + 10y = -8.$
   $3y - 5x = 5.$
10. $4x + 3y = 14.$
    $x - 6y = -1.$
11. $2x + y = 1.$
    $3x - 2y = 1\frac{3}{4}.$
12. $2x + 5y = 20.$
    $3x - 2y = -46.$

13. $6x + 15y + 6 = 0.$
$14y + x = 22.$

14. $18x + 10y = 60.$
$12x - 15y = 105.$

15. $5x + 3y = 29.$
$3x + 7y = 7.$

16. $9H - 10K = 122.$
$4H - 15K = 12.$

17. $y - 12x = 0.$
$2y - 9x = 7\frac{1}{2}.$

18. $\frac{x}{2} + \frac{y}{7} = 9.$
$\frac{x}{7} - \frac{y}{2} + 5 = 0.$

19. $.2x - .3y = 2.$
$.4x + .6y = 5.2.$

20. Construct and solve two equations when
$x = 4$ and $y = 5.$

108. Elimination by Substitution. — This method appears to have been first used by Sir Isaac Newton.

HISTORICAL NOTE

Sir Isaac Newton was not only one of the greatest mathematicians the world has ever produced, but he was also one of its greatest scientists and philosophers.

He was born in Lincolnshire, England, in 1642. His father died shortly before Newton was born, but left the widow comfortably situated, so that Isaac had an opportunity to acquire an education. As a child he was very frail, however, and gave little promise of any unusual intellectual attainments.

When he was 18 years old, he entered Cambridge University, and it was here that the real genius of the man began to manifest itself. He mastered Euclid (equivalent to what we now call plane geometry) in a single reading, and in fact regarded the whole subject as self-evident. After that he read all the mathematical literature available with the greatest relish, and soon began to add brilliant discoveries of his own.

Among the greatest discoveries of this wonderful man, we may mention the *binomial theorem*, by means of which any binomial may be raised to any required power, and the *laws of motion* and of *universal gravitation*. The influence of these discoveries on the sciences, especially physics and astronomy,

was very great. Newton appears to have been the first to use the substitution method of elimination.

Newton was a man of the greatest modesty, so that he hesitated to publish his discoveries, and it was only by the insistence of his friends and pupils that they were given to the world. Perhaps his greatest achievement was a method now called *calculus*. As a result of his reluctance to tell of his discoveries this was not known to the outside world for a long time.

Newton was for many years Professor of Mathematics at Cambridge University, and afterwards held important government positions. He died in London in 1727 and is buried in Westminster Abbey.

Newton's greatest discoveries are contained in the "Principia" published in 1687, which, in the words of the great scientist and mathematician Laplace, has "a lasting preëminence over all other productions of the human mind."

Elimination by substitution may be shown in the solution of the following examples:

Illustrative example I:

Solve and check: $2x + 5y = 8.$
$x - 10y = 9.$

Solution and explanation:

$2x + 5y = 8.$
$x - 10y = 9.$
$x = 10y + 9.$
$2(10y + 9) + 5y = 8.$
$20y + 18 + 5y = 8.$
$20y + 5y = 8 - 18.$
$25y = -10.$
$y = -\frac{2}{5}$
$x + 4 = 9.$
$x = 5.$
$*\therefore x = 5.$
$y = -\frac{2}{5}.$

Check: $10 - 2 = 8.$
$5 + 4 = 9.$

The second equation may be solved for x in terms of y by adding $+ 10y$ to both members of the equation. We obtain $x = 10y + 9$. This cannot be considered an answer, how-

* The sign $\therefore$ meaning *therefore*, is common in both algebra and geometry.

ever, as it contains the unknown quantity y, and our final answer must contain only known quantities.

Since $x = 10y + 9$ in this system, we may substitute $10y + 9$ in place of x in the first equation and get $2(10y + 9) + 5y = 8$. This equation does not contain x. The x was eliminated by substituting for it an equal expression containing no letter but y. Hence, we have eliminated *x by substitution.*

Removing the parenthesis and solving the equation $2(10y + 9) + 5y = 8$, we obtain $y = -\frac{2}{5}$. Then by substituting $y = -\frac{2}{5}$ in the second equation, we obtain $x + 4 = 9$, and find that $x = 5$. We now check these values in the original equations. Hence, the solution of the system is

$$x = 5,\ y = -\tfrac{2}{5}.$$

Illustrative example II:

Solve
$$5x + 2y = -2. \quad (1)$$
$$4x - 3y = 26. \quad (2)$$

Solution: Solving the second equation for the value of x, we obtain:

$$x = \frac{3y + 26}{4}. \quad (3)$$

$5\left(\frac{3y + 26}{4}\right) + 2y = -2.$ Substituting $\frac{3y + 26}{4}$ in place of x in the first equation.

$\frac{15y + 130}{4} + 2y = -2.$

$15y + 130 + 8y = -8.$ Clearing of fractions.

$23y = -138.$

$y = -6.$ Solving for the value of y.

$5x - 12 = -2.$ Substituting the value of y in the first equation. You may substitute in (3) if you prefer.

$5x = 10.$

$x = 2.$

Checking in the original equations,

$$10 - 12 = -2.$$
$$8 + 18 = 26.$$

Therefore, $x = 2$, $y = -6$, are the roots of the equation.

Rule. — (1) *Solve* ***one*** *of the equations for one of the unknown quantities expressed in terms of the other, thus getting a third equation* (3).

(2) *Substitute the expression obtained in place of the quantity for which it stands in the* ***other*** *equation.*

(3) *Solve that equation.*

(4) *Substitute that value in* (3).

(5) *Check in* (1) *and* (2).

Oral Exercises

1. Solve $x + y = 15$, when $x = y$.
2. Solve $2x + y = 12$, if $x = y$.
3. Solve
$$x = 2y.$$
$$x - y = 6.$$
4. Solve
$$x = 3y.$$
$$x + 2y = 10.$$
5. Solve
$$x = \tfrac{1}{3}y.$$
$$3x + y = 8.$$

Exercises

Solve by substitution:

1. $3x + 10y = 2.$
$x = 6 + 2y.$
2. $y = 2x - 1.$
$x + 4y = 23.$
3. $x = 3y - 23.$
$2x + 3y = -1.$
4. $5x + 3y = 0.$
$y = 8x - 29.$
5. $y = 4x + 2.$
$10x + 2y = 13.$
6. $x = -6y - 1.$
$3y - 2x = 12.$

7. $x = 1 - 4y.$
$2x + y = -6\frac{3}{4}.$

8. $3x + 4y = 2.$
$y = 4x - 9.$

9. $x = 34 - 5y.$
$7x - 6y = 33.$

10. $K = 60 - 8H.$
$7H = 9 + 10K.$

11. $2x - y = 5.$
$5x - 2y = 14.$

12. $6x + 10y = 19.$
$2y = 3x.$

Hint: Substitute the value of $3x$ directly.

13. $x = 4.$
$5x + 6y = 50.$

14. $2x - 7y = 9.$
$5x + 3y = 2.$

15. $9x - 8y = 5.$
$15x + 12y = 2.$

16. $4m - 3n = 0.$
$2m + 5n = 26.$

17. $5s - 3t = 36.$
$5t - 7s = -56.$

18. $2x + 7y = -\frac{1}{2}.$
$3x + 10y = -1.$

19. $3x + 2y = 15.$
$4y - \frac{1}{2}x = 17.$

20. $\dfrac{2x - 3y}{7} + \dfrac{3x + 4y}{6} - 3 = 0.$
$x = y - 4.$

Selection Exercise

Select the correct pair of values of x and y in each system.

1. $x + 2y = 8.$
$3x - y = 17.$
$x = 1, 3, 5, 6, -4.$
$y = -2, 2, 3, 1, 0.$

2. $3x - 4y = 12.$
$x + 5y = -15.$
$x = -11, \frac{1}{11}, 0, 11.$
$y = -\frac{4}{5}, \frac{1}{5}, -3, 3.$

3. $x = 3y + 2.$
$y = 3x + 2.$
$x = 1, 2, -1, 0.$
$y = 3, 1, -1, 1.$

4. $2x + 3y = 5.$
$x - y = 0.$
$x = 0, 1, 2, 1.$
$y = 0, 1, 3, -2.$

5. $6x + y = 16.$
$4x = 14 - y.$
$x = 1, 10, -1.$
$y = 10, 1, 4.$

Systems Involving Decimals

Solve and check as directed by your teacher:

1. $.3x + .4y = 3.4.$
   $.5x - .3y = 1.8.$
2. $1.8x + 3y = 0.$
   $2.4x - 4.2y = 49.2.$
3. $4.2x - 6.4y = 39.2.$
   $1.5x + .2y = 31.4.$
4. $1.25x - .6y = 30.$
   $.75x + .45y = 78.75.$
5. $.6x + .75y = -1.8.$
   $1.5x + 6.5y = 14.$
6. $.7x + .8y = -9.6.$
   $1.8x - 1.5y = 18.$
7. $12.5x - 6.4y = 1.45.$
   $15x - 20.5y = -7.875.$
8. $5.4x + 3.2y = 33.3.$
   $7.5x - 10.8y = -22.35.$
9. $1.5x - 6.5y = -5.5.$
   $7.5x + 3.45y = 8.45.$
10. $5.4x + 8.75y = 5.025.$
    $-8.2x + 10.5y = -28.05.$

109. Comparison of Methods. — Any system of simultaneous linear equations can be solved by either of the methods which have been explained. Frequently, however, there is a distinct advantage in using one method rather than another. In the exercise which follows you should study the system carefully and decide in each case which method of elimination is the easiest. Clear of fractions first.

Exercises

Solve each of the following systems of equations:

1. $\frac{x}{4} - \frac{y}{7} = 2.$
   $\frac{x}{5} - \frac{y}{3} = -3.$
2. $x = 3y.$
   $\frac{x}{3} + y = 34.$
3. $\frac{x}{7} + \frac{y}{5} = \frac{10}{7}.$
   $x + \frac{y}{3} = \frac{14}{3}.$
4. $\frac{x - 5y}{2} = x - 3.$
   $x - 2y = 3\frac{1}{5}.$

5. $\frac{5x}{6} + \frac{y}{4} = 7.$

$\frac{2x}{3} - \frac{y}{8} = 3.$

6. $\frac{x}{5} = \frac{y}{4}.$

$3x + \frac{1}{2}y = 17.$

7. $\frac{20x + 9}{7} - \frac{y}{7} = 1.$

$5x - 5y = -29.$

8. $\frac{x}{2} - \frac{y}{3} = 1.$

$\frac{x}{4} - \frac{y}{9} = 1.$

9. $\frac{x - y}{2} = \frac{25}{6} - \frac{x + y}{3}.$

$\frac{x + y - 9}{2} - \frac{y - x - 6}{3} = 0.$

10. $\frac{m + 1}{3} = \frac{2 - n}{2} - \frac{5}{6}.$

$\frac{3m - 1}{2} = \frac{n - 3}{3} - \frac{1}{4}.$

11. $\frac{1}{2}(y + 1) + (x - 5) = -4.$

$\frac{x + 1}{3} + y = 3.$

12. $\frac{m}{2} - 1 = \frac{n}{3}.$

$\frac{m}{4} - \frac{n}{9} = 1.$

13. $\frac{x}{2} - \frac{y}{3} = 0,$ and $\frac{y}{2} - \frac{x}{4} = 2.$

14. $\frac{14x + 3}{5} - \frac{8(y + 4)}{17} = 0.$

$\frac{10x - 2}{3} - \frac{4y + 3}{4} = 0.$

110. Fractional Equations using $\frac{1}{x}$ and $\frac{1}{y}$. In the type of fractional equations in which the *unknown quantities form the denominators,* the elimination should usually be performed by addition or subtraction *before* clearing of fractions.

Illustrative example:

Solve for values of x and y.

$$\frac{1}{x} - \frac{2}{y} = 7.$$

$$\frac{3}{x} + \frac{4}{y} = 1.$$

Solution:

$$\frac{2}{x} - \frac{4}{y} = 14$$ Multiplying the 1st equation by 2.

$$\frac{3}{x} + \frac{4}{y} = 1$$ Second equation.

$$\frac{5}{x} = 15$$ By adding.

$$5 = 15\,x.$$ By clearing of fractions.

$$x = \tfrac{1}{3}.$$

$$3 - \frac{2}{y} = 7.$$ Substituting the value of x in the first equation.

Therefore, $3\,y - 2 = 7\,y.$

$$-4\,y = 2.$$

$$y = -\tfrac{1}{2}.$$

Check in original equations.

Exercises

Solve and check:

1. $\frac{1}{x} + \frac{1}{y} = \frac{3}{4}.$
   $\frac{1}{x} - \frac{1}{y} = \frac{1}{4}.$

2. $\frac{1}{x} - \frac{1}{y} = \frac{1}{4}.$
   $\frac{1}{x} + \frac{1}{y} = \frac{3}{8}.$

3. $\frac{4}{x} - \frac{3}{y} = 2.$
   $\frac{10}{x} - \frac{9}{y} = 3.$

4. $\frac{2}{x} - \frac{3}{y} = 1.$
   $\frac{4}{x} - \frac{9}{y} = -7.$

5. $\frac{3}{x} - \frac{5}{y} = 3.$
   $\frac{5}{x} + \frac{5}{y} = 1.$

6. $\frac{21}{y} + 2\,x = -15.$
   $\frac{7}{y} + x = -8.$

7. $2\,x + \frac{3}{y} = 17.$
   $5\,x - \frac{1}{y} = 34.$

8. $\frac{4}{x} + \frac{3}{y} = 12.$
   $\frac{2}{x} - \frac{5}{y} = -7.$

9. $\frac{1}{x} - \frac{4}{y} = 8.$
$\frac{3}{x} + \frac{2}{y} = 3.$

10. $\frac{1}{x} + \frac{2}{y} = \frac{1}{10}.$
$\frac{3}{x} - \frac{4}{y} = 2\frac{3}{10}.$

11. Jones and Brown can do a piece of work in 10 days. They can do the same work if Jones works 12 days and Brown works 6 days. How many days would it require each to do it?

Hint: See example 3, page 279.

12. If one pipe discharges water into a tank for 4 hours, another pipe could fill it in $2\frac{2}{3}$ hours. If the first pipe flows for 2 hours, the other pipe can fill it in 4 hours. How long would it take each pipe alone to fill the tank?

111. Literal Simultaneous Equations may be solved by the same methods as numerical ones. If the value obtained for the first unknown quantity is at all complicated, it is frequently best to obtain the second unknown, not by substitution of this result, but by a new elimination in the original equations.

Illustrative example:

Solve $\left.\begin{aligned} ax + by &= m \\ cx + dy &= n \end{aligned}\right\}$ for the values of x and y.

Solution:

$adx + bdy = md$ Multiplying the 1st equation by d.

$cbx + bdy = nb$ Multiplying the 2d equation by b.

$(ad - cb)x = md - nb$ Subtracting.

$x = \frac{md - nb}{ad - cb}.$ Dividing by the quantity $(ad - cb)$.

Again, $acx + bcy = mc$ Multiplying the 1st equation by c and the 2d by a.

$acx + ady = na$

$(bc - ad)y = mc - na$ Subtracting.

Check. $y = \dfrac{mc - na}{bc - ad}$. Dividing by the quantity $(bc - ad)$.

Exercises

Solve each of the following systems for x and y:

1. $3x - y = 10b.$
   $4x + 9y = 3b.$

2. $3x + 4y = 6m.$
   $\dfrac{x}{m} - 2 = \dfrac{3y}{4m}.$

3. $lx + 2my = n.$
   $2lx - my = -3.$

4. $2cx + 2y = 6.$
   $y - 3 = \dfrac{x}{c}.$

5. $2x - 3y = a - b.$
   $3x - 2y = a + b.$

6. $\dfrac{2x}{5} + \dfrac{2y}{3} = \dfrac{6a}{5}.$
   $\dfrac{x}{2} + \dfrac{y}{2} = a.$

7. $ax + by = k.$
   $cx - dy = l.$

8. $\dfrac{a}{b + y} = \dfrac{b}{a - x}.$
   $\dfrac{a}{b + x} = \dfrac{b}{a - y}.$

9. $a^2x + b^2y = c^2.$
   $p^2x + q^2y = r^2.$

10. $\dfrac{a}{x} + \dfrac{b}{y} = 1.$
    $\dfrac{c}{x} + \dfrac{d}{y} = 2.$

11. $\dfrac{1}{x} - \dfrac{1}{y} = r.$
    $\dfrac{1}{x} + \dfrac{1}{y} = s.$

12. $\dfrac{x - y + 1}{x - y - 1} = a.$
    $\dfrac{x + y + 1}{x + y - 1} = b.$

13. $(a - b)x - y = 0.$
    $x + (a - b)y = 0.$

14. $\dfrac{x}{a} + \dfrac{y}{b} = k.$
    $\dfrac{x}{a} - \dfrac{y}{b} = l.$

15. $5.7x + 2y = 10h.$
    $.19x + .4y = 2h.$

16. Solve each of the physics formulas $s = \frac{1}{2}at^2$ and $F = Ma$ for a, then equate your results and find a formula from which a is missing.

17. In a similar manner get rid of (eliminate) g in the physics formulas $v = gt$ and $s = \frac{1}{2}gt^2$ and obtain a formula for v. For s.

Exercises on Systems of Equations

Algebraic Shorthand

Unless other letters are required in a problem, use x and y and let x be the greater number.

Express the following in algebraic shorthand:

1. The sum of two numbers is 10.

2. The difference of two numbers is 4.

3. One number is 6 more than another number.

4. The sum of the squares of two numbers is 25.

5. The difference of the squares of two numbers is 64.

6. The product of two numbers is 72.

7. The quotient of two numbers is 3.

8. Five times one number less another number is 19.

9. Six times the smaller number subtracted from 8 times the larger equals 16.

10. The fraction whose denominator is x and numerator is y equals one half.

11. The value of n nickels and d dimes is 75 cents.

12. If 12 is added to the numerator of the fraction $\frac{y}{x}$, the new fraction is $\frac{x}{y}$.

13. The annual income (simple interest) on d dollars at 6% is x dollars.

14. The yearly income on a 4% investment of d dollars is y dollars.

15. The result of adding 6 to the denominator of $\frac{y}{x}$ is $\frac{1}{3}$.

16. The result of x divided by y gives a quotient of 10 and a remainder of 2.

17. The perimeter p of a rectangle whose length is x and width y.

18. The area (A) of the above rectangle.

19. If a certain fraction is $\frac{x}{y}$, what is the meaning of $\frac{x+5}{y-4}$? How does the fraction $\frac{x}{y}$ change when (**a**) 5 is subtracted from the numerator? (**b**) 5 is subtracted from the denominator?

20. The volume of a rectangular solid x feet long, y feet wide, and 6 feet high is 480 cubic feet.

21. John can shingle a roof in 5 days and Jim can do it in 6 days. What part of the work can each do in 1 day?

22. Henry can do a job in x days and George can do it in y days. What does $\frac{1}{x}+\frac{1}{y}$ mean?

23. The quotient of the difference of two numbers divided by half their sum is one ninth.

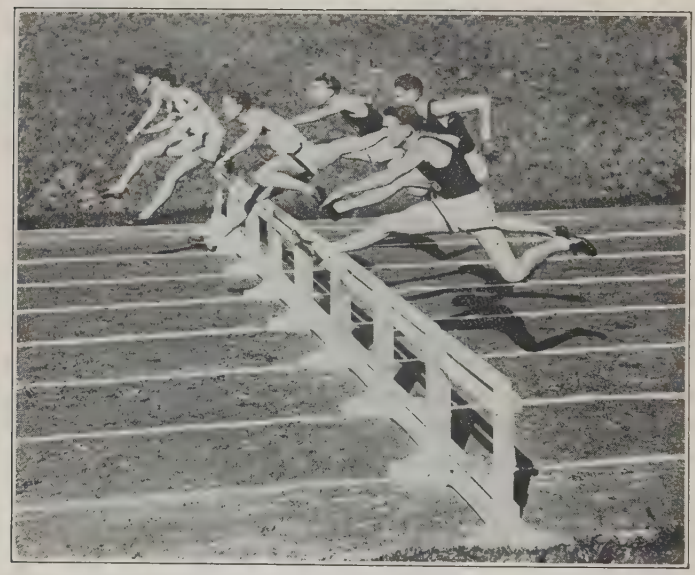
24. Express the number n if the tens' digit is t and the units' digit is u.

25. The sum of the squares of two numbers exceeds 75 by 25.

26. The perimeter of a rectangle becomes 108 when the length and width are each trebled.

27. The perimeter of a rectangle becomes 80 if the length is doubled and the width is quadrupled.

28. The perimeter of an isosceles triangle becomes 48 if each of the two equal sides x are doubled and the base remains constant.



Running the 220 Hurdles

29. One boy runs the 220-yard hurdles in x seconds. A second boy runs them in y seconds. How far will both run in 1 second?

30. If one pipe stream will fill a tank in x hours and another will fill it in y hours, what part of the tank will be filled in 1 hour when both pipe streams are running together?

Problems Using Two Unknowns

Number Problems

Solve each of the following problems by a system of linear equations:

1. Find two numbers whose sum is 114 and difference 58.

2. The difference of two numbers is 17 and their sum is 33. Find the numbers.

3. The sum of two numbers is 1200. Three times the smaller plus twice the larger equals 2704. What are the numbers?

4. One number is four times another. The sum of the numbers is 140. Find the numbers.

5. Six times a larger number exceeds two times the smaller by 156. The difference of the numbers is 18. Find them.

6. Separate 154 so that six times the smaller equals the larger number.

7. Nine times the larger number is 186 more than seven times the smaller. Their sum is 42. Find them.

8. The number of inches in the length of a rectangle is 13 more than the number of inches in the width. Their sum is 43. What are the numbers?

9. Separate 600 into two numbers so that 25% of the larger added to 12.5% of the smaller equals 135.

10. Find two numbers whose sum is a and whose difference is b.

11. A and B have d dollars together. If A gives B n dollars, they then have equal amounts. How much has each?

Digit Problems

1. 625 means $600 + 20 + 5$. What does 768 mean?

2. Express a number which has t tens and u units.

3. Express a number with h hundreds, t tens, u units. State the sum of these digits. The product. A number with these digits reversed.

4. Find a two-digit number whose tens' digit is four times the units' digit if the sum of the digits is 10.

5. The sum of the digits of a two-digit number is 12. The tens' digit is 2 more than the units' digit. Find the number.

6. Three times the tens' digit exceeds the units' digit by 5. The sum of the digits is 11. What is the number?

7. A two-digit number is four times the sum of the digits. If the digits are interchanged, the result is 4 plus twenty times the tens' digit. Find the number.

8. If a two-digit number be divided by the units' digit, the result is 16. The sum of the digits is 15. Find the number.

9. If 18 be added to the number, the resulting number is expressed by the same digits in reverse order. The sum of the digits is 10. What is the number?

Hint: Let x = the digits in the tens' place.

y = the digit in the units' place.

Then $10\,x + y$ = the number.

$$\left.\begin{aligned} x + y &= 10 \\ 10\,x + y + 18 &= 10\,y + x \end{aligned}\right\}$$ Complete the solution.

10. In a three-digit number the tens' digit is 5 more than the units' digit. The hundreds' digit is four times the units' digit. The sum of its digits is 17. Find the number.

11. If a certain number of two digits be divided by the sum of its digits, the quotient is 2 and the remainder 8. Three times the given number is 2 greater than the number expressed by the same digits in reverse order. What is the number?

12. A certain number of two digits equals 4 times the sum of its digits, while the number made by reversing the order of the digits is 27 greater than the given number. What is the number?

Fractional Problems

1. The difference of two numbers is 187 and their quotient is 12. Find the numbers.

2. $\frac{2}{3}$ of one number is 2 less than half of another number, and the sum of the numbers is 67. What are they?

3. Two numbers are in the ratio of 2 to 3. Their sum is 30. Find the numbers.

4. Two numbers are in the ratio of 3 to 5. 4 times the larger added to 7 times the smaller equals 82. Find them.

5. The numerator and denominator of a fraction are in the ratio of 5 to 6. If 4 is added to the numerator and 16 is added to the denominator, the resulting fraction equals $\frac{3}{5}$. What is the original fraction?

Hint: $\frac{x}{y} = \frac{5}{6}$. $\frac{x+4}{y+16} = \frac{3}{5}$.

6. There is a fraction whose value is $\frac{2}{3}$. If two be subtracted from the numerator and added to the denominator, its value becomes $\frac{1}{2}$. Find the fraction.

7. The sum of the numerator and denominator of a certain fraction is 28. If 2 be added to the numerator and subtracted from the denominator, the value of the resulting fraction is $2\frac{1}{2}$. Find the original fraction.

8. The sum of two fractions is $\frac{5}{6}$, and the smaller is $\frac{2}{3}$ of the larger. What are they?

9. The sum of two numbers is m, and the quotient of those two numbers is n. Find the numbers.

10. Two books cost together d dollars and one cost c cents more than the other. Find the cost of each.

11. If A gives B c dollars, A will have twice as much as B. If B gives A d dollars, A will have 6 times as much as B. How much has each?

Coin Problems

1. 37 coins consisting of nickels and dimes amounted to \$2.30. How many coins of each kind were there?

2. A purse contained 22 coins consisting of quarters and dimes. If their total value is \$4.45, how many coins of each kind were there?

3. I bought 80 stamps consisting of 1-cent and 2-cent stamps. If they cost me \$1.30, how many were there of each kind?

4. I bought 130 stamps consisting of 5-cent, 2-cent, and 1-cent stamps. There were twice as many 2-cent stamps as 5-cent ones. The total cost was \$1.90. How many did I buy of each kind?

Business Problems

1. A man buys two houses for \$15,000. He pays \$650 more for one than he does for the other. Find the value of each house.

2. The admission to an entertainment was 50¢ for adults and 25¢ for children; the total receipts from 150 tickets were \$62.50. Find (a) the number of adults admitted; (b) the number of children admitted.

3. One airplane cost 4 times as much as a certain automobile. Two such airplanes cost \$6000 more than six of the automobiles. Find the cost of each.

4. On one occasion a man buys 3 tons of hard coal and 2 tons of soft coal, paying \$51; on another occasion he buys 2 tons of hard coal and 6 tons of soft coal, paying \$62. What is the price of each?

5. One automobile costs the same as 5 smaller automobiles of a similar model. Two of the former costs \$1200 more than 8 of the latter. Find the cost of each.

ON THE ALERT

6. A man sold cows and calves, 13 animals in all, for \$475. For the cows he received \$50 each and for the calves \$15 each. How many of each did he sell?

7. At Hop Lee's laundry 3 collars and 4 cuffs cost 25¢ while 10 collars and 6 cuffs cost 54¢. Find the cost of laundering each.

8. Two partners, Williams and Jones, wish to divide their profits in the relation of 3 to 4. If they make \$8400, what would each receive?

Hint: $\frac{x}{y} = \frac{3}{4}$ might be considered as one equation.

9. Two partners A and B agree to divide their yearly profits of \$16,000 in the ratio of their investments which are \$45,000 by A and \$75,000 by B. How much will each receive?

Business Investment Problems

1. A man had \$8600 to invest. He invested part of it in bonds, at par, paying 5% and the remainder in a mortgage paying 6%. The income from both investments was \$454. How much was invested at each rate? See page 171.

2. A part of \$5000 was invested at $4\frac{1}{2}\%$ and a part at $5\frac{1}{2}\%$. The $4\frac{1}{2}\%$ investment yields \$75 more each year than the $5\frac{1}{2}\%$ investment. How much is invested at each rate?

3. A man invests a certain amount at 6% and \$4000 more than that at 5%. His annual income is \$750. How much has he invested at each rate?

4. One fourth of the insurance money left a widow she invests at 5%, and the remainder at 6%. The latter produces \$650 a year more than the former. Her annual income was \$1150. What was the amount of the insurance?

Mixture Problems

1. 5 pounds of tea and 8 pounds of coffee cost \$4.34, while 10 pounds of tea and 3 pounds of coffee cost \$5.04. What is the price of each per pound?

2. How many pounds of 50-cent tea and how many pounds of 65-cent tea must be mixed to make a mixture of 24 pounds worth 60 cents a pound?

3. How many pounds of 36-cent coffee and how many pounds of 22-cent coffee must be mixed to make a mixture of 50 pounds to sell for \$15.62?

4. How many pounds of 15-cent rice and how many pounds of 20-cent rice must be mixed to make 50 pounds of 17-cent rice?

5. How many pounds of 30-cent candy and how many pounds of 80-cent candy must be mixed to make 90 pounds to sell at 60 cents a pound?

6. How many quarts of 60-cent cream and how many quarts of 12-cent milk must be mixed to make a mixture of 36 quarts worth \$14.40?

7. Milk testing 3% butter fat is to be mixed with cream testing 28% butter fat. How many quarts of each must be mixed to make 10 gallons which shall test 18% butter fat?

Motion Problems

1. A man rowed up a river 10 miles in 5 hours and back in $2\frac{1}{2}$ hours. Find the rate of the current and his rate of rowing in still water.

Hint: Let x = rate in still water.
y = rate of current.

	Rate	Time	Distance
Downstream .	$x + y$	$2\frac{1}{2}$	10
Upstream . .	$x - y$	5	10

$$2\tfrac{1}{2}(x + y) = 10.$$
$$5(x - y) = 10.$$

2. A steamer goes downstream 60 miles in 4 hours. The return trip takes two hours longer. What is the rate of the current and the rate of the boat in still water?

3. Flying with the wind a bird was able to make 150 miles an hour, but flying against a wind of only half as great velocity it could make only 30 miles an hour. Find the velocity of the wind and the bird's rate of flying when there was no wind.

4. If the velocity of a cannon report with the wind is 350 yards per second and 330 yards a second against the wind, what is the velocity of the wind?

5. A boy rows 8 miles with the current in 1 hour and 4 minutes, and returns in $2\frac{2}{7}$ hours. At what rate would he row in still water? What is the rate of the current?

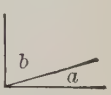
6. Flying against the wind an airplane takes 6 hours to travel 480 miles but returns with the wind in 4 hours. Find the velocity of the wind and the speed of the plane in still air.

Miscellaneous Problems

1. The width of a rectangle is $\frac{2}{3}$ of its length and its perimeter is 80 yards. Find its dimensions.

2. The perimeter of a rectangle is 300 feet. The length is 30 feet more than the width. Find the dimensions.

3. Two angles a and b in the figure are said to be complementary when their sum is equal to 90°. If one of these angles is 10° greater than three times the other, how many degrees in each angle?



4. The force P necessary to lift a weight W by means of a certain machine is given by the formula

$$P = a + bW,$$

where a and b are constants depending on the amount of friction in the machine. If a force of 7 pounds will raise a weight of 20 pounds, and a force of 13 pounds will raise a weight of 50 pounds, what force is necessary to raise a weight of 40 pounds?

Hint: First find the values of a and b in that machine.

5. Three years ago Mary was three times as old as John. Three years hence she will be twice as old. Find the present age of each.

6. Half of Henry's age added to $\frac{1}{3}$ of Frank's age is 11 years. Six years from now the sum of their ages will be 40 years. How old is each now?

7. A and B together can do a piece of work in $3\frac{3}{7}$ days, and B requires two days longer than A to do the whole work. Find the time required by each to do the work.

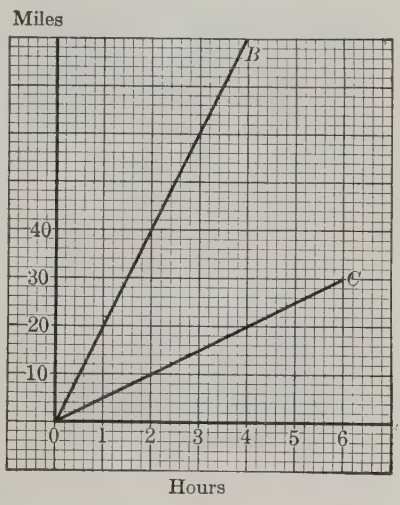
8. John and his father can do a piece of work in $4\frac{1}{2}$ days. After they have worked 3 days the father falls ill, and John finishes the task in 6 days. How many days does each require to do the entire work?

9. The sum of the reciprocals of two numbers is $\frac{10}{21}$ and the difference of their reciprocals is $\frac{4}{21}$. What are the numbers?

10. The sum of two angles is 73° and their difference is 37°. How many degrees in each angle?

11. If 6 pounds of butter and 4 dozen eggs cost $\$m$ and 4 pounds of butter and 3 dozen eggs cost $\$n$, what is the price of a pound of butter and of a dozen eggs?

112. Problems Solved by Graphs. — Problems of many kinds, especially those involving the idea of rate, can often be solved conveniently by a graph.

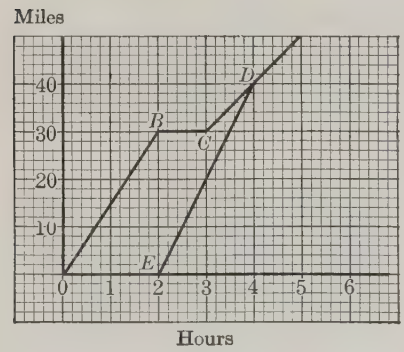


Mark off units of time on the horizontal axis, — in this case hours. On the vertical axis mark off units of distance. In this case let one large square represent 10 miles. Now suppose we want to represent a rate of 20 miles per hour. In 0 hours at this rate one will go 0 miles, in 1 hour 20 miles, in 2 hours

40 miles, etc. Joining these points, we have the straight line OB. In like manner a rate of 5 miles an hour is represented by the line OC. This line does not represent the path of the traveler, but the slant of the line represents the rate. The steeper the line the greater the rate represented.

Illustrative example I:

John leaves a certain point on his bicycle at the rate of 15 miles an hour. Two hours later he has an accident which delays him 1 hour, after which he proceeds at 10 miles an hour. Two hours after John started, James started to overtake him, riding at the rate of 20 miles an hour. How long and how far must he ride?

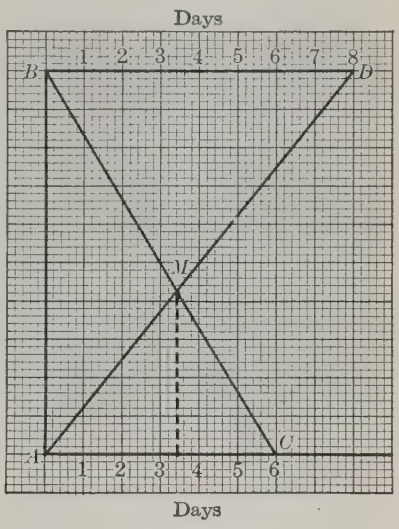


Solution: Draw OB to represent the rate of John and continue it until it reaches the ordinate corresponding to 2 hours. He was then delayed an hour, hence time went on, but distance did not change. The line BC becomes therefore horizontal at this point. Then CD is drawn in a direction to indicate 10 miles an hour, and continued indefinitely.

The fact that James started two hours later than John is indicated by starting the line ED two units to the right from the origin, and the direction of the line ED represents a rate of 20 miles an hour. The intersection D shows that John's rate and James's rate have caused them to reach a distance of 40 miles simultaneously. Hence James has overtaken John. From the axes we read a distance of 40 miles and a time of 2 hours for James and 4 hours for John.

Illustrative example II:

A can do a piece of work in 8 days and B in 6 days. In how many days can they do the work if they work together?



Solution: A convenient distance AB is chosen on the vertical axis to represent the magnitude of the entire task. The line AD is drawn to represent a rate of work which accomplishes the task in 8 days, and BC is drawn to represent a rate which accomplishes it in 6 days. Directly beneath the point of intersection M we read from the graph that both together can do the work in slightly more than 3.4 days. The algebraic solution shows that the exact result is $3\frac{3}{7}$ days. When solving by graphs, you should be able to read the answer correctly to the nearest tenth.

Exercises

Solve each of the following problems by a graph:

1. A and B start from the same point and travel in the same direction, one at 20 miles an hour and the other at 24 miles an hour. In how many hours will they be 28 miles apart?

2. Two men, A and B, start at the same time on a journey from M to N, forty miles away. A travels 20 miles an hour, and B 16 miles an hour. When A reaches his destination, he

immediately returns at the same rate. How far from N will he meet B?

3. A man travels for two hours at 15 miles an hour, then for one hour at 10 miles an hour, and then for three hours at 30 miles an hour. What uniform rate would have enabled him to make the same distance in the same time?

4. Henry starts to walk to town at 3 miles an hour. After two hours he rests one half hour, and then walks on at 2 miles an hour. Stephen starts 3 hours after Henry and walks 4 miles an hour. After how many hours will he overtake Henry? If the distance to town is 20 miles, how much sooner will Stephen arrive than Henry?

5. A starts from P toward Q at 8 A.M., at the rate of 10 miles an hour. At 9 A.M. B starts from Q toward P at 15 miles an hour. If the distance from P to Q is 60 miles, how far from Q will they meet?

6. B works 4 days on a piece of work which he can do in 9 days. Then he is joined by C, who could do the whole work in 8 days. If C works until the task is finished, how many days does he work?

7. Mr. Brown can plow a field in 4 days. Mr. Jones and he together can plow the field in $2\frac{1}{2}$ days. How long will it take Mr. Jones to plow the field alone?

8. Two pipes can fill a tank, one in two hours and 40 minutes, the other in 3 hours and 40 minutes. How long will it take to fill the tank if both pipes are open?

9. Two boys run around a quarter mile track, one making the circuit in $1\frac{1}{4}$ minutes and the other in $1\frac{1}{2}$ minutes. If they start together, how many times around will each have run before they are together again?

At Work in the Garden

10. One boy can hoe a garden in 6 hours, another in 4 hours. How long will it take them to hoe the garden if they work together?

True-False Test

Copy the numbers of the following eight examples and mark them with a *t* or *f*.

1. $x = 2$, and $y = 1$ are the roots of $\left\{\begin{matrix} x - 2y = 0 \\ 2x + 3y = 7 \end{matrix}\right\}$.

2. The system $\left\{\begin{matrix} x + y = 6 \\ x - y = 6 \end{matrix}\right\}$ cannot be solved.

3. $x = 3$ and $y = 2$ are the only values that satisfy the equation $x + y = 5$.

4. The equation $2x + 5y = 10$ has an unlimited number of sets of roots.

5. The equation $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$ is true when $x = \frac{1}{3}$ and $y = \frac{1}{2}$.

6. The system $\left\{\begin{matrix} x+y=6 \\ x+y=3 \end{matrix}\right\}$ can be solved graphically.

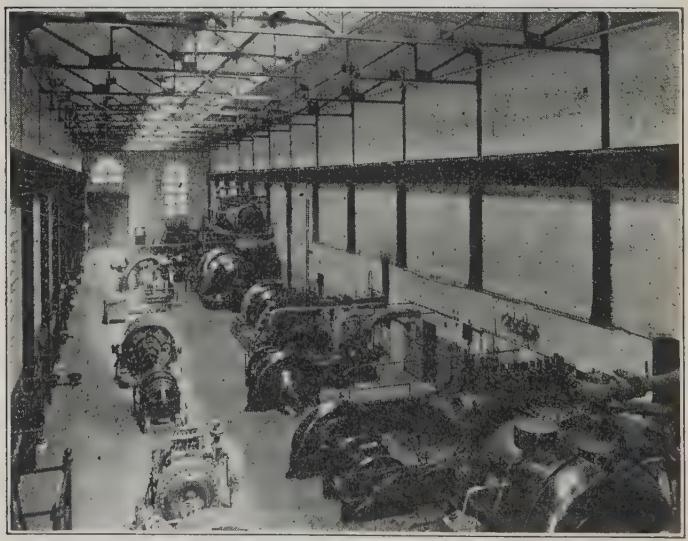
7. The system $\left\{\begin{matrix} \frac{1}{x}+\frac{1}{y}=\frac{7}{12} \\ \frac{2}{x}+\frac{3}{y}=\frac{17}{12} \end{matrix}\right\}$ has fractional roots.

8. The roots of the system $\left\{\begin{matrix} 2x-7y=0 \\ 3x+5y=0 \end{matrix}\right\}$ are $x=0$, $y=0$.

Review

1. Find the value of the following as indicated:

(a) $51^2 = (50+1)^2$. (b) $29 \cdot 31 = (30-1)(30+1)$.



The Flywheel Steadies the Flow of Power

2. The kinetic energy of a flywheel rim is given by the formula $K.E. = \frac{Wv^2}{2g}$, where W equals the weight in tons. Find W if $g = 32$, $v = 60$, $K.E. = 480$.

3. A pupil obtains a term mark of 90 in algebra, 80 in French, and 78 in English. What must he obtain in Latin to maintain an average of 81 in his four subjects?

4. Define and illustrate:

(a) Indeterminate equation. (c) Systems of equations.
(b) Linear equation. (d) Elimination.

5. Solve $F = Ma$ for M.

6. Solve $\frac{s}{C} = \frac{a}{360}$ for s; for a.

7. What is the area of a triangle whose base is $(x + y)$ and whose altitude is $(x - y)$?

8. The difference of two numbers is d and their sum is s. Find the value of the two numbers.

9. If x and y represent two numbers, write in words the meaning of the equations $\left\{\begin{matrix} x + y = a \\ x - y = b \end{matrix}\right\}$ and find the value of the two numbers.

10. If the cost of an article is c, the profit based on the selling price is $p\%$, and the cost of doing business computed on the selling price is $b\%$, find the selling price S in terms of the other letters.

11. Solve the formula written in example 10 for the value of c in terms of the other letters.

12. If I gain c cents on an article which cost me d dollars (including all overhead expenses, such as heat, light, telephone, postage, etc.), what per cent do I make on the cost? on the selling price?

In examples 13–32 tell which have errors and which have none. Also give the correct results of the former.

13. $\frac{x^3}{x^3} = 0.$ **14.** $\frac{5x + 1}{5} = x + 1.$

15. $\frac{3}{4}(a+b)=\frac{3\,a}{4}+\frac{3\,b}{4}$.

16. $(x-2)^2=x^2+4$.

17. $\frac{x^2-y^2}{x+y}=x-y$.

18. $2^3=6$.

19. $2\,x+3\,y=6\,xy$.

20. $0(a+b)=a+b$.

21. $(2\,x)^2=2\,x^2$.

22. $\frac{3\,x-y}{y-3\,x}=-1$.

23. If $\frac{x}{3}+\frac{x}{4}=1$, $4\,x+3\,x=1$.

24. $\frac{1}{3}+\frac{1}{4}=\frac{4}{7}$.

25. $3\,x^2+7\,x-6=(x-3)(3\,x+2)$.

26. $1^3=1$.

27. $\frac{x}{y-z}=-\frac{x}{z-y}$.

28. $23-3\times4=20\times4=80$.

29. $xyzw=wxzy$.

30. $5\,xy-(-5\,xy)=0$.

31. $\frac{x+k}{y+k}=\frac{x}{y}$.

32. If $\frac{3}{x}=1$, $x=3$.

33. Express in algebraic shorthand:

(a) The sum of the squares of $4\,x$ and $5\,y$.

(b) The difference of the squares of $4\,x$ and $5\,y$.

(c) The square of the sum of $4\,x$ and $5\,y$.

34. If sound travels approximately 1130 feet per second, make a formula for the distance it would travel in t seconds.

35. The telephone day rate from New York to Albany is 90 cents for the first 3 minutes and 30 cents for each additional minute. Let n equal the number of additional minutes. Make a table of values for $n = 0, 1, 2, 3, 4, 5$, and then make a formula for any number of additional minutes. How does the rate change as n decreases?

36. Graph the correct formula for example 35.

CHAPTER XI

POWERS AND ROOTS

Now that we have studied equations through systems and fractional linear equations, there is just one more step to prepare us for the climax of our study, the solution of quadratic equations. This step is the mastery of powers and roots. Powers have already been treated (section 33); the roots we are now to discuss are not the roots of equations, but square and cube roots and so forth.

113. Finding Powers of a quantity is a shorthand method of multiplication. It may be shown by the following

Illustrative examples:

I. $(4x^2)^2 = 4x^2 \cdot 4x^2 = 16x^4.$

II. $(2x^2y^4)^3 = 2x^2y^4 \cdot 2x^2y^4 \cdot 2x^2y^4 = 8x^6y^{12}.$

III. $(-3x^5)^2 = (-3x^5)(-3x^5) = +9x^{10}.$

By studying these illustrations we may infer the following

Rule. — *To raise an algebraic quantity to a given power*

(1) *Raise the numerical coefficient to the required power.*

(2) *Multiply each of the exponents of the given quantity by the exponent of the power to which it is to be raised.*

The law of exponents in finding powers may be expressed in symbols thus,

$$(x^m)^n = x^{mn}.$$

Oral Exercises

Perform the indicated operation in each case:

1. $(3x)^2$.
2. $(-3x)^2$.
3. $(4y^2z)^2$.
4. $(-4y^2z)^2$.
5. $(6x^3)^2$.
6. $(-6x^3)^2$.
7. $(3mn^3y)^2$.
8. $(-3mn^3y)^2$.
9. $(5ab^4)^2$.
10. $(-5ab^4)^2$.
11. $(12x^5)^2$.
12. $(-12x^5)^2$.
13. $(13xyz)^2$.
14. $(-13xyz)^2$.
15. $(25a^2b^2)^2$.
16. $(-25a^2b^2)^2$.
17. $(15HK)^2$.
18. $(-15HK)^2$.
19. $(4x^2)^3$.
20. $(3ab^2)^3$.
21. $(-2x)^3$.
22. $(-3m^2n)^3$.
23. $(5m^3x^2)^3$.
24. $(-2x^2yz)^3$.
25. $(3x^2)^4$.
26. $(-2ms)^4$.
27. $(4x)^4$.
28. $(2x^2)^5$.
29. $(-y^2)^5$.
30. $(-2x^3y)^5$.

114. Finding Roots of a Quantity. — The square root of a quantity is one of the two equal factors of that quantity. The cube root of a quantity is one of the three equal factors of the quantity. A similar definition holds for higher roots.

Since $(3x)^2 = 9x^2$ and $(-3x)^2$ also equals $9x^2$, it follows from the above definition that both $3x$ and $-3x$ are square roots of $9x^2$. We may therefore state the following

Principle. — *Every algebraic number or expression has two square roots, numerically equal, but opposite in sign.*

Since $(2x)^3 = 8x^3$ and $(-2x)^3 = -8x^3$ we see that the cube root of $8x^3$ is $2x$ and the cube root of $-8x^3$ is $-2x$.

A quantity, therefore, has but one real cube root, which has the same sign as the quantity.

115. Radical Signs. — The extraction of a root is often indicated by the sign $\sqrt{}$, called the *radical sign*. When written as above it indicates square root. Thus, $\sqrt{16\,x^4y^6}$ is read "the square root of $16\,x^4y^6$" and means that we are to find the square root of that quantity. Similarly, $\sqrt[3]{8\,x^6}$ means "the cube root of $8\,x^6$" and $\sqrt[4]{81\,x^{12}}$ means "the fourth root of $81\,x^{12}$." The small figure at the upper left-hand corner of the radical sign is called the *index* of the root, and shows what root is to be found. In the square root sign the index 2 is implied.

HISTORICAL NOTE

The earliest instance of the use of the radical sign to indicate a root to be extracted occurs in an arithmetical work by the Frenchman Nicholas Chuquet. This work was known as Le Triparty and was written in 1484. Chuquet also used indices in the modern sense to indicate the root to be taken.

116. Law of Signs in Finding Roots. — The principle governing signs in the extraction of square root, which was explained in section 114, holds for any root whose index is an even number, and the principle explained for cube root holds for any root whose index is an odd number.

We can now state the following

Rule. — *To extract the root of a monomial,* (1) *find by trial the root of the numerical coefficient.*

(2) *Divide the exponent of each literal factor by the index of the root.*

(3) *Annex these literal factors with their new exponents to the root of the numerical coefficient.*

(4) *Prefix the* $\pm$ *sign if the index is even. Prefix the given sign if the index is odd.*

The sign $\pm$ is read "plus or minus."

In general $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

Illustrative examples:

I. The square root of $4\,x^4y^{10}$ is $\pm\,2\,x^2y^5$.

II. The cube root of $-\,27\,x^6y^9$ is $-\,3\,x^2y^3$.

III. The fourth root of $16\,x^8y^{20}$ is $\pm\,2\,x^2y^5$.

Oral Exercises

What is the square root of each of the following expressions?

1. $64\,x^8$.
2. $36\,m^2n^6$.
3. $16\,a^4b^{10}$.
4. $144\,a^8b^4$.
5. $9\,m^2n^2x^6$.
6. $25\,a^8b^{12}$.
7. $81\,x^{16}z^{20}$.
8. $121\,a^{10}m^4n^2$.
9. $49\,x^{14}y^{12}z^{20}$.
10. $100\,s^2t^4$.
11. $25(a-b)^2s^6$.
12. $\frac{1}{4}(x-2\,y)^4$.
13. $\frac{1}{16}\,s^2t^6$.
14. $\frac{4}{9}\,a^4t^6$.
15. $\frac{9}{16}\,a^8b^2$.
16. $\frac{25}{81}\,m^{10}n^{16}$.
17. $.09\,x^4y^2$.
18. $.16\,a^8b^4$.
19. $(H+K)^2(M-N)^4$.
20. $64\,x^2s^{10}(a-b)^2$.
21. $36\,x^{20}y^{12}$.
22. $256\,x^{2n}$.

In examples 23–29, use only positive square roots.

23. $\sqrt{25}+5+\sqrt{4}=\,?$
24. $\sqrt{4\cdot 25}=\,?$
25. $\sqrt{4+21}+\sqrt{16}=\,?$
26. $\dfrac{2\sqrt{81}-\sqrt{16}}{7}=\,?$
27. What is the value of $\sqrt{x}-\sqrt{y}$ if $x=25$ and $y=16$?
28. What is the value of $4\sqrt{x}+3\sqrt{y}$ if $x=16$ and $y=9$?
29. (a) $\sqrt{16+9}=\,?$
    (b) $\sqrt{16}+\sqrt{9}=\,?$
    (c) Are these two results equal?
30. What is the square root of $\dfrac{.25\,x^2}{.36\,y^4}$?

State the cube root of each of the following:

31. $27\,x^9y^{18}$. **33.** $-8\,a^3b^{15}$.

32. $64\,m^3n^{12}$. **34.** $125\,m^{15}n^{30}$.

35. $-27\,x^{21}(a-b)^3$.

What is the fourth root of

36. $16\,x^4$? **38.** $256\,m^{20}$?

37. $81\,y^8x^{12}$? **39.** $625\,x^8$.

State the fifth root of

40. $-32\,x^{10}y^{15}$. **41.** $243(a-b)^5c^{15}$.

117. Square Root of a Trinomial. — A binomial expression is never a perfect square, unless it contains radicals, and its square root can, therefore, not be found exactly.

A trinomial square is, however, a very common algebraic expression. Such a square is formed by squaring a binomial, as explained in section 78. By the rule there stated $(3\,x-4\,y)^2 = 9\,x^2 - 24\,xy + 16\,y^2$. The square root of $9\,x^2 - 24\,xy + 16\,y^2$ is, therefore, $3\,x - 4\,y$. $-(3\,x-4\,y)$ is also a square root of $9\,x^2 - 24\,xy + 16\,y^2$, so that we ought to inclose $3\,x - 4\,y$ in a parenthesis preceded by the $\pm$ sign; thus, $\pm\,(3\,x-4\,y)$.

The square root of a trinomial square like the above may always be found mentally by the following

Rule. — (1) *Take the square root of the first term of the trinomial and the square root of the last term.*

(2) *Make the first of these terms positive, and give the second the sign of the middle term of the trinomial.*

(3) *Inclose the resulting binomial in a parenthesis preceded by the* $\pm$ *sign.*

In applying the above rule the trinomial must first be arranged in descending powers of some common letter.

Oral Exercises

State the square roots of each of the following trinomials:

1. $x^2 + 6x + 9$.
2. $x^2 - 12x + 36$.
3. $x^2 - 8xy + 16y^2$.
4. $4x^2 - 12xy + 9y^2$.
5. $25x^2 - 10x + 1$.
6. $25a^2 + 30ab + 9b^2$.
7. $100m^2 - 40mn + 4n^2$.
8. $36x^2 + 12xy + y^2$.
9. $x^2 - 14xz + 49z^2$.
10. $16x^2 - 16x + 4$.
11. $s^2 - 20s + 100$.
12. $81 + 54x + 9x^2$.
13. $16x^2 + 8 + \frac{1}{x^2}$.
14. $a^2 + \frac{4a}{b} + \frac{4}{b^2}$.
15. $.04x^2 + .12xy + .09y^2$.

118. Square Roots of Numbers. — In our study of arithmetic we learned how to extract the square root of numbers. We can extend its use here.

Illustrative example:

Let us extract the square root of 138,384. The number must first be pointed off in groups of two figures each, beginning at the decimal point, which, in the case of an integer, is at the right. The number of figures in the root agrees with the number of groups so obtained.

Solution: The largest square less than 13 is 9. $\sqrt{9} = 3$.

Subtract, bringing down one group

Annex 0 to this root 3 (making 30) and multiply by 2 for a trial divisor. Trial divisor $= 2 \cdot 30 = 60$. $483 \div 60 = 8$. However, as the trial divisor must later be increased, 8 will be found

$$\begin{array}{r} 3 \\ \hline 13'83'84. \\ 3^2 = 9 \\ \hline 4\,83 \end{array}$$

too large. We therefore take 7 as the second figure. Complete the divisor by adding this second figure: $60 + 7 = 67$. Multiplying the complete divisor by the second figure of the root, we have

Subtract, bringing down final group

Annex 0 to the new root 37 and multiply by 2 for the trial divisor:

$2 \cdot 370 = 740$

$1484 \div 740 = 2$

Complete divisor $= 740 + 2 =$

$$\begin{array}{r} 3\ 7\ 2. \\ \hline 13'83'84. \\ 3^2 = 9 \\ \hline 4\ 83 \\ 7 \cdot 67 = 4\ 69 \\ \hline 14\ 84 \\ 742 \\ 2 \cdot 742 = 14\ 84 \end{array}$$

Exercises

In like manner extract the square root of

1. 7396. 2. 15129. 3. 183184. 4. 813604.

5. 328.515625 to the nearest hundredth.

Solution:

$$\begin{array}{r} 1\ 8.\ 1\ 2\ 5 \\ \hline \sqrt{3'28.51'56'25} \\ 1 \\ \hline \text{Trial Divisor } 2 \mid 2\ 28 \\ \text{Complete Divisor } 28 \mid 2\ 24 \\ \hline \text{T. D. } 36 \mid 4\ 51 \\ \text{C. D. } 361 \mid 3\ 61 \\ \hline \text{T. D. } 362 \mid 90\ 56 \\ \text{C. D. } 3622 \mid 72\ 44 \\ \hline \text{T. D. } 3624 \mid 18\ 12\ 25 \\ \text{C. D. } 36245 \mid 18\ 12\ 25 \end{array}$$

$\therefore$ The square root to nearest hundredth is $\pm$ 18.13.

NOTE. If result is desired to nearest tenth, find root to two decimals and then adjust the second decimal, 1 more if your second decimal of root is 5 or more.

Find to the nearest tenth the square root of

6. 3.0697.	**9.** .385672.	**12.** 296172.
7. 458.49.	**10.** 1.68.	**13.** 25.06.
8. 16.9744.	**11.** 4287.	**14.** 286.33.

15. $\frac{3}{8}$. *Hint:* First change the fraction to a decimal.

16. $\frac{5}{16}$. **17.** $3\frac{5}{9}$. **18.** $8\frac{2}{3}$. **19.** $12\frac{5}{8}$.

Find to the *nearest hundredth* the square root of

20. 394.1728.	**22.** 1194.6.	**24.** $22\frac{1}{3}$.
21. 16.72.	**23.** $\frac{5}{11}$.	**25.** 18.0625.

26. Find to the nearest thousandth the square root: (a) of 3; (b) of 2. Memorize both your answers.

27. Find to the nearest tenth the number of rods in the side of a square field which contains 8 acres.

Hint: 1 acre = 160 square rods.

119. Finding Roots by a Table. — The easiest way to find the square root or cube root of small numbers is by a table. The table on page 341 shows the roots and the squares or cubes of numbers up to 100.

Illustrative examples:

1. Find the square root of 39 by the use of the table.

Solution: Look down the first column until you find 39. Then read across until under the square root column and find 6.244. *Ans.*

2. Find the square root of 4672.

Solution: From the table we find the two perfect squares nearest to 4672; they are 4624 and 4761, the squares of 68 and 69.

Now we know that our desired root lies between 68 and 69. 4672 is $\frac{48}{137}$ of the difference between 4624 and 4761. (4761 − 4624 = 137. 4672 − 4624 = 48.) Therefore, the

Table of Powers and Roots

No.	Squares	Cubes	Square Root	Cube Root	No.	Squares	Cubes	Square Root	Cube Root
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	2.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4.913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

square root of 4672 must be 68 plus $\frac{48}{137}$ of the difference between 68 and 69. $(69 - 68 = 1.)$ $\frac{48}{137}$ of $1 = .3503$.

68 plus .3503 gives us the required root, 68.3503.

This is accurate to two decimal places. Usually this method will give the square root to the nearest hundredth.

Check by another method.

3. Find the cube root of 41.

4. See if you can find the approximate cube root of 569125 by a method similar to example 2. Check by cubing your answer.

By the use of the table find to the nearest hundredth the square root of

1. 19.	**4.** 8.	**7.** 15.	**10.** 55.
2. 39.	**5.** 67.	**8.** 57.	**11.** 21.
3. 97.	**6.** 70.	**9.** 32.	**12.** 46.

Find the results of the following:

13. $\sqrt[3]{20} - \sqrt[3]{10}$. **14.** $\sqrt{50} - \sqrt{25}$. **15.** $\sqrt{64} + \sqrt{8}$.

120. Approximate Measurement. — The figures in the preceding table are accurate to the nearest thousandth. Could you measure the length of your desk correctly to a thousandth of an inch? To a hundredth of an inch? If you were very careful, you might measure it correctly to a tenth of an inch. Boys and girls with good eyes and good rulers should be able to do that. Suppose you found your desk to be 26.7 inches long. Do you think its real length would be exactly that? Could you be sure that it was not really 26.72 or possibly 26.69?

When we say that the length of the desk is 26.7 inches we mean that that is its length to the *nearest tenth.* If it were possible for us to measure it correctly to hundredths, it might

be found to be anywhere between 26.65 and 26.74 inclusive. Similarly if you measured the width of the desk you would, perhaps, obtain 14.3 inches, and this might mean anything between 14.25 and 14.34 inclusive.

Now suppose you were asked to find the area of this desk in square inches. The exact product of 26.7 and 14.3 is 381.81, so you see that if the length were *exactly* 26.7 inches and the width were *exactly* 14.3 inches, the area would be *exactly* 381.81 square inches. However, since 26.7 and 14.3 are not exact do you think it reasonable to claim that the area is *exactly* 381.81 sq. in., or that it is reasonable to attempt to express the area to hundredths of a square inch? Isn't it more sensible to express this result to the nearest tenth and call the answer 381.8 square inches? For some purposes it would be sufficient to express the answer to the nearest square inch, calling the result 382 square inches. Few of you could do better than that in an actual measurement problem.

Exercises

1. Measure the length and width of your algebra to the nearest tenth of an inch. Find its area to the nearest tenth of a square inch.

2. Express each of the following numbers (**a**) to the nearest integer. (**b**) to the nearest tenth. (**c**) to the nearest hundredth.

(**a**) 168.085. (**c**) 6.457. (**e**) .763.
(**b**) 81.962. (**d**) 11.928. (**f**) 2.526.

3. Multiply each of the following approximate numbers, expressing the result in each case to the nearest tenth.

(**a**) 81.7×32.8. (**c**) 16.5×38.8. (**e**) 28.8×11.92.
(**b**) 29.3×11.1. (**d**) 19.7×86.35. (**f**) 56.61×92.83.

An Attractive Front Walk

4. Using a yard stick measure the length of the front walk leading to your school to the nearest foot.

5. Measure the length and width of your schoolroom to the nearest inch.

6. Express the products of the following, rounding off the answer to the nearest unit.

(a) 16.5×91.7.
(b) 86.2×64.1.
(c) 82.8×12.7.
(d) 95.2×108.6.

121. Rounding Off Numbers. — It is often unnecessary to give exact values, especially when dealing with large numbers. Most business people use what are called *round numbers* or approximate values. For example if the exact amount is \$5,975,876, it is usually called \$6,000,000 approximately. In population problems such approximations are often used: the population of a city may be called 1,000,000 when it really is a little below or a little above 1,000,000.

Numbers are usually rounded off to the nearest hundred, thousand, or million. However, exact figures are often

necessary, as in business tabulations. Again in baseball we may need to carry the work out to four or possibly five decimal places in order to determine the leading team or leading batter of the league. For example, one man might have a batting average of .39435 and another .39436. Which would lead? You will have to use your own common sense whether you are to work with absolute exactness or with round numbers when you get out in the world of business.

122. Approximate Results Involving Square Root. — In measuring anything we have seen that a measured length is only approximately correct. On the other hand, indicated roots are always exact. In the following calculations it is to be understood that the number under the radical sign is exact, while the other numbers are measured numbers and therefore only approximate.

By use of the table, using the nearest tenth find the results of the following:

1. $5.4\sqrt{3}$.
2. $6.5\sqrt{2}$.
3. $7\sqrt[3]{80}$.
4. $6\sqrt[3]{90}$.
5. $\sqrt{69} \div 3.1$.
6. $\sqrt[3]{34} \div 4.6$.

Give these results in round numbers.

Completion Test

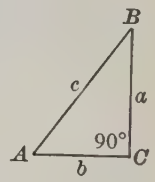
1. If equals are multiplied by equals, the results are . . .
2. Minus times plus gives . . .
3. The distance divided by the rate gives . . .
4. There are . . . even roots of a positive number.
5. There is . . . odd root of a negative number and it is always . . .
6. Exponents are added in . . .
7. We may check an equation by . . . the . . . in the . . . equation.

8. We add a negative number and a positive number by . . . them, and giving the result the . . . of the larger absolute number.

9. The absolute value of + 7 is the same as that of . . . namely . . .

10. If we subtract a negative number from a positive number the remainder is . . . than the minuend.

123. Use of Square Root in Solving Right Triangles. — The adjoining triangle ABC is called a right triangle because it has a right angle at C. This angle at $C = 90°$. In all such triangles the square of the side opposite the angle of 90° (called the *hypotenuse*) is always equal to the sum of the squares of the other two sides (called the *legs*) or, stated as an equation,



$$c^2 = a^2 + b^2,$$

or, extracting the square root of both members,

$$c = \sqrt{a^2 + b^2}.$$

HISTORICAL NOTE

This important mathematical principle may be stated thus: *In a right triangle the sum of the squares of the legs is equal to the square of the hypotenuse.*

The discovery of this principle was made by Pythagoras, a Greek mathematician, who was born in Samos in 569 B.C. After having studied at Thebes and Memphis and at various places in Asia Minor, he finally returned to Samos, where he established a school of mathematics and philosophy. Not being very successful here, he went to Tarentum and later to Croten in southern Italy. Here he gathered around him a large number of followers and established a school which became very influential, in both educational and political circles.

The members of the Pythagorean society had all things in common, held their discoveries secret, and lived very simple and austere lives. One member of the society, Hippasus, was drowned by his fellow members for having boasted about his discoveries.

The high character of the Pythagoreans, the nature of their philosophy, and the secret character of their organization enabled them to gain much political influence, and made them many enemies. Finally, in 500 B.C., Pythagoras and many of his followers were murdered by a mob.

PYTHAGORAS

This principle, called for Pythagoras the "Pythagorean principle," is one of the most used propositions in all mathematics, especially with practical and applied problems relating to heights and distances. Some of its applications are illustrated in many of the following problems.

Exercises

1. Find the value of c in a right triangle whose altitude $(a) = 8$, and whose base $(b) = 6$. What was the side c named in your study of arithmetic?

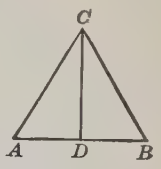
2. Solve the equation $c^2 = a^2 + b^2$ for a^2. For a. Then find the value of a if $c = 5$ and $b = 4$.

3. Solve the equation $c^2 = a^2 + b^2$ for b^2. For b. Then find the value of b if $c = 5$ and $a = 3$.

4. The top of a ladder 25 feet long is even with the top of a wall a feet high. The foot of the ladder is 13.5 feet away from the foot of the wall. How high is the wall?

5. A railway incline is 5 in each 150 (*i.e.* 5 feet high for each 150 feet of slant height, or when $c = 150$, $a = 5$). Find the projected or horizontal length.

6. A perpendicular is drawn from the vertex to the base of an equilateral (all sides equal) triangle each of whose sides is 4 inches. Find the length of the perpendicular CD.



7. The side of a square is 2 inches. Find the length of the straight line which joins two opposite corners of the square.

8. If it is 90 feet from home base to first base on a baseball diamond, and 90 feet from first base to second base, find the shortest distance from home base to second base.

9. Find the hypotenuse and the area of a right triangle whose legs are 42 feet and 26 feet, respectively.

10. Find the base of a right triangle whose hypotenuse (c) $= 33$ feet, and whose altitude (a) $= 28$ feet.
Find the area.

11. Find the diagonal (the line joining the opposite corners) of a rectangle, if the base is 70 feet and the altitude (a) is 12 feet.

12. The diagonal (c) of a certain rectangle is twice the length of the altitude (a) of the rectangle. If the base (b) of that rectangle is 5 inches, find the altitude and the diagonal. Find the area of the rectangle.

13. The velocity of the flow of water under a head is denoted by $v = c\sqrt{2\,gh}$. Find v when

$$c = .62$$
$$g = 32.$$
$$h = 15.7.$$

14. The velocity of a falling body is denoted by the formula $v = \sqrt{2\,gs}$. What is the velocity of an object which falls from a height (s) of 555 feet (the height of the Washington monument in Washington) when it strikes the ground?

Hint: $g = 32.16$.

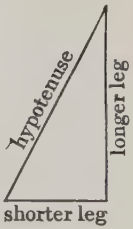
Washington Monument

15. Find to the nearest tenth of an inch the side of a square whose area is 600 square inches.

16. Find to the nearest tenth of a foot the side of a square whose area is equal to the area of a rectangle with a base of 50 feet and an altitude of 20 feet.

17. The altitude of a right triangle is $1\frac{1}{3}$ times the base. The hypotenuse is 15. Find the base and the altitude.

Hint: If b = base, a = altitude, c = hypotenuse, $a^2 + b^2 = c^2$.



18. In a right triangle the longer leg is 2 inches more than the shorter leg and 2 inches less than the hypotenuse. Find each of the sides of the triangle.

19. One leg of a right triangle is 3 feet longer than the other leg, and the area of the triangle is 5 square feet.

Find the length of each leg.

124. Irrational Quantities. — The numbers we have studied up till now have all been rational. A *rational number* is a

positive or negative integer or the indicated quotient of such integers as $3, -7, \frac{2}{3}, .3, -\frac{8}{5}, \sqrt{4}$.

We are now going to take up another kind of number, called irrational. An *irrational number* is a number which cannot be exactly expressed by any integer or any fraction.

$\sqrt{3}, \sqrt[3]{5}, \sqrt{1+\sqrt{2}}$ are irrational numbers.

A *radical* is an indicated root of any quantity. Ex. $b\sqrt[n]{a}$.

The *radicand* is the quantity under the radical sign.

A *surd* is the simplest type of an irrational number, that in which the radicand is rational.

Thus, $\sqrt{5}, \sqrt[3]{9}, \sqrt[4]{6}, \sqrt{\frac{1}{2}}$ are surds, while the more complicated expression $\sqrt[3]{3-2\sqrt{5}}$ is not a surd.

The *order* of a radical is indicated by its index. When the index is not expressed, it is understood to be 2.

Thus, $\sqrt{5}, 2\sqrt{11}, \sqrt{15}$ are radicals of the second order, while $\sqrt[3]{6}, 5\sqrt[3]{19}$ are of the third order.

Similar radicals are radicals of the same order, having the same radicand, as $2\sqrt{3}, 6\sqrt{3}, -10\sqrt{3}$; $b\sqrt[3]{2}, c\sqrt[3]{2}$, etc.

Oral Exercise

1. State which of the following are rational, and which are irrational:

$2, 6\frac{3}{4}, \sqrt{7}, \sqrt{9}, 11, \frac{3}{4}, \sqrt[3]{6}, \sqrt[2]{5-\sqrt{2}}, \sqrt{1}, \sqrt[3]{\frac{1}{8}}, 17.$

2. State the radicand in each of the following radicals:

$3\sqrt{11}, \quad 2\sqrt[3]{26}, \quad \sqrt[3]{104}, \quad \sqrt[5]{3-\sqrt{6}}, \quad 6-\sqrt{5}, \quad 3\sqrt[4]{17},$

$2\sqrt{5-3\sqrt[3]{2}}, \qquad \sqrt{171}.$

3. State the order of the above radicals.

4. State which of the above radicals are also surds.

5. State the index of the following radicals:

$\sqrt{19}, \sqrt[6]{64}, \sqrt[5]{x^4}, \sqrt[7]{2^5}, \sqrt{3}, \sqrt[3]{125}, \sqrt{2}.$

Which are rational?

125. Addition of Radicals. — Only similar radicals can be *added* or *subtracted.* This is done according to the following

Rule. — *To add similar radicals, add the coefficients and annex the common radicand.*

Illustrative example I:

Add $5\sqrt{2}, 3\sqrt{2}, 8\sqrt{2}, -6\sqrt{2}$.

Solution: We obtain $(5 + 3 + 8 - 6)\sqrt{2} = 10\sqrt{2}$.

Illustrative example II:

Add $a\sqrt{3} + 2b\sqrt{3} - 5\sqrt{3} + 2\sqrt{3}$.

Solution:

We obtain $(a + 2b - 5 + 2)\sqrt{3} = (a + 2b - 3)\sqrt{3}$.

Illustrative example III:

Add $2\sqrt[3]{x}, 5\sqrt[3]{x}, -6\sqrt[3]{x}$.

Solution: We obtain $(2 + 5 - 6)\sqrt[3]{x} = \sqrt[3]{x}$.

Exercises

Combine the following radicals:

1. $4\sqrt{2} + 3\sqrt{2} + 10\sqrt{2} - 6\sqrt{2} + \sqrt{2} - 8\sqrt{2}$.
2. $6\sqrt{3} + 5\sqrt{3} - 2\sqrt{3} + 16\sqrt{3} - 5\sqrt{3}$.
3. $5\sqrt{x} - 2\sqrt{x} + 11\sqrt{x} - 6\sqrt{x} + \sqrt{x}$.
4. $5\sqrt{6} - 10\sqrt{6} - 3\sqrt{6} - 11\sqrt{6} + \sqrt{6}$.
5. $2\sqrt{10} + 3\sqrt{10} - 6\sqrt{10} + 4\sqrt{10}$.
6. $2a\sqrt{x} + 3a\sqrt{x} - 7a\sqrt{x} - 10a\sqrt{x}$.
7. $3\sqrt[4]{2} - 6\sqrt[4]{2} - 8\sqrt[4]{2} + 10\sqrt[4]{2} - \sqrt[4]{2}$.
8. $a\sqrt[3]{6} + 2b\sqrt[3]{6} + 3c\sqrt[3]{6} - 4d\sqrt[3]{6}$.
9. $a\sqrt{b} + c\sqrt{b} - m\sqrt{b}$.
10. $a\sqrt{5} + 3b\sqrt{5} + 10c\sqrt{5} + 10c\sqrt{5} - 8\sqrt{5}$.

126. Subtraction of Radicals. — The subtraction of radicals is as simple as addition. For it, we have the following

Rule. — *To subtract similar radicals, subtract the coefficients and annex the common radicand.*

Illustrative example I:

From $3\sqrt{2\,a}$ subtract $-6\sqrt{2\,a}$.

Solution:

Subtracting -6 from 3 we obtain $+9$.

Hence $3\sqrt{2\,a}-(-6\sqrt{2\,a})=9\sqrt{2\,a}$.

Illustrative example II:

From $3\sqrt{2}+6\sqrt{5}$ subtract $2\sqrt{2}-5\sqrt{5}$.

Solution:

$$\begin{array}{rl} 3\sqrt{2}+\ 6\sqrt{5} & \text{Or}\quad (3\sqrt{2}+6\sqrt{5})-(2\sqrt{2}-5\sqrt{5}) \\ \underline{2\sqrt{2}-\ 5\sqrt{5}} & \qquad =3\sqrt{2}+6\sqrt{5}-2\sqrt{2}+5\sqrt{5} \\ \sqrt{2}+11\sqrt{5} & \qquad =\sqrt{2}+11\sqrt{5}. \end{array}$$

Illustrative example III:

From $m\sqrt[3]{a}$ subtract $n\sqrt[3]{a}$.

Solution: $\quad m\sqrt[3]{a}-n\sqrt[3]{a}=(m-n)\sqrt[3]{a}$.

Exercises

1. From the radical in the odd column subtract that in the even column in order to the right.

$5\sqrt{2}$	$3\sqrt{2}$	$-3\sqrt{5}$	$6\sqrt{5}$	$6\sqrt{7}$	$-3\sqrt{7}$	$6\sqrt[3]{2}$	$-3\sqrt[3]{2}$
$8\sqrt{3}$	$-2\sqrt{3}$	$5\sqrt{10}$	$8\sqrt{10}$	$-2\sqrt{7}$	$-10\sqrt{7}$	$a\sqrt{5}$	$c\sqrt{5}$
$10\sqrt{6}$	$4\sqrt{6}$	$-2\sqrt{3}$	$-3\sqrt{3}$	$9\sqrt{3}$	$-2\sqrt{3}$	$5\,a\sqrt[3]{2}$	$6\,b\sqrt[3]{2}$

2. From $3\sqrt{6}-2\sqrt{5}+3\sqrt{2}$ subtract $5\sqrt{6}+2\sqrt{5}-\sqrt{2}$.

3. From $4\sqrt{2}-7\sqrt{3}+9\sqrt{7}$ subtract $6\sqrt{3}+2\sqrt{7}$.

4. From $5\sqrt[3]{2} - 4\sqrt[3]{6} + 5\sqrt[3]{10}$ subtract $4\sqrt[3]{2} - 4\sqrt[3]{6} + \sqrt[3]{10}$.

5. From $6\sqrt{a} - 10\sqrt{b} + \sqrt{c}$ subtract $8\sqrt{c} + 11\sqrt{b} - \sqrt{a}$.

6. What must be added to $3\sqrt{6} + 7\sqrt{5} - 1$ to produce $8\sqrt{6} - 4\sqrt{5} + 8$?

7. By how much does $3\sqrt{x} + 5\sqrt{y} - \sqrt{z}$ exceed $2\sqrt{x} + 7\sqrt{y} - 2\sqrt{z}$?

8. By how much is $8\sqrt[3]{a} + 10\sqrt[3]{b} + 16\sqrt[3]{c}$ greater than $2\sqrt[3]{a} - 15\sqrt[3]{b} - 8\sqrt[3]{c}$?

127. Transformation of Radicals. — We may often change the *form* of a radical without changing its *value*. Such a change is called a *transformation*. One of the commonest cases of transformation is the reduction of the radicand to its smallest possible integral value. The following illustrative examples show how this transformation is accomplished.

I. $\sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$.

II. $\sqrt{300} = \sqrt{100 \cdot 3} = 10\sqrt{3}$.

III. $\sqrt[3]{56} = \sqrt[3]{8 \cdot 7} = 2\sqrt[3]{7}$.

IV. $\sqrt{3\,a^3b^2} = \sqrt{a^2b^2 \cdot 3\,a} = ab\sqrt{3\,a}$.

From these transformations we may deduce the following

Rule. — (1) *Factor the radicand into two factors, one of which is the largest possible power whose degree is equal to the order of the radical under consideration.*

(2) *Extract the indicated root of that factor and write it as a coefficient of the radical whose radicand is the other factor, multiplying it by the coefficient which already exists, if any is written.*

Illustrative example:

Make the radicand of $2\sqrt{48}$ as small as possible by a transformation.

Solution: Since this radical is of the second order we must find a factor of 48 which is an exact second power, or a perfect square. This factor is 16, the square of 4.

We now have $2\sqrt{48} = 2\sqrt{16 \cdot 3} = 2 \cdot 4\sqrt{3} = 8\sqrt{3}$.

Exercises

Transform the following radicals so as to make their radicands as small as possible.

1. $\sqrt{32}$.
2. $\sqrt{8}$.
3. $\sqrt{72}$.
4. $\sqrt{50}$.
5. $\sqrt{162}$.
6. $\sqrt{200}$.
7. $-\sqrt{162}$.
8. $2\sqrt{18}$.
9. $4\sqrt{45}$.
10. $4\sqrt{50}$.
11. $3\sqrt{20}$.
12. $7\sqrt{63}$.
13. $3\sqrt{28}$.
14. $11\sqrt{27}$.
15. $-2\sqrt{32}$.
16. $-2\sqrt{40}$.
17. $\sqrt{3\,a^2}$.
18. $ab\sqrt{9\,b^3}$.
19. $-b\sqrt{8\,b^2}$.
20. $10\sqrt{90\,a}$.
21. $-3\sqrt{8\,a^2}$.
22. $\frac{1}{2}\sqrt{48}$.
23. $\frac{2}{3}\sqrt{200}$.
24. $\frac{1}{5}\sqrt{75\,a^2}$.
25. $\frac{1}{b}\sqrt{7\,b^2c}$.
26. $\frac{c}{a}\sqrt{9\,a^2c^2x}$.
27. $\frac{2}{x}\sqrt{x^2yz}$.
28. $\frac{1}{2}\sqrt{4\,a^4x^3}$.
29. $6\sqrt{x^6y^3z}$.
30. $\sqrt[3]{81}$.
31. $\sqrt[3]{54}$.
32. $\sqrt[3]{162}$.
33. $\sqrt[3]{16}$.
34. $\sqrt[4]{32}$.
35. $\sqrt[4]{162}$.
36. $-\sqrt[3]{192}$.
37. $3\sqrt[3]{16}$.
38. $3\sqrt[4]{64}$.
39. $3\sqrt[5]{64}$.
40. $2\sqrt[3]{72}$.
41. $5\sqrt[3]{9\,a^3}$.
42. $2\,a\sqrt[3]{54\,a^3b^4}$.
43. $\frac{1}{2}\sqrt[3]{54}$.
44. $\frac{1}{2}\sqrt[3]{8\,x^3y^4}$.

128. Transformation of a Radical Making Its Radicand an Integer (Whole Number). — This method of transformation may be made clear by the following

Illustrative examples:

I. Transform $\sqrt{\frac{3}{8}}$ so as to make the radicand a whole number.

Solution: $\sqrt{\frac{3}{8}} = \sqrt{\frac{3}{8} \cdot \frac{2}{2}} = \sqrt{\frac{6}{16}} = \sqrt{\frac{1}{16} \cdot 6} = \frac{1}{4}\sqrt{6}.$

Here we have multiplied both numerator and denominator of the given fraction by 2. This does not change the value of the given fraction, but makes the denominator a perfect square. The fraction $\frac{6}{16}$ is then factored and the radical transformed according to the rule of section 127.

II. $\sqrt{1\frac{1}{3}} = \sqrt{\frac{4}{3} \cdot \frac{3}{3}} = \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{9} \cdot 3} = \frac{2}{3}\sqrt{3}.$

III. $\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{3}{4} \cdot \frac{2}{2}} = \sqrt[3]{\frac{6}{8}} = \sqrt[3]{\frac{1}{8} \cdot 6} = \frac{1}{2}\sqrt[3]{6}.$

Rule. — (1) *If the radicand is a mixed number, reduce it to a common fraction.*

(2) *Multiply both numerator and denominator of this fraction by the smallest number which will make the denominator a perfect power of a degree equal to the order of the radical.*

(3) *Factor and transform by the rule of the preceding section.*

Exercises

Transform each of the following radicals so the radicand shall be a positive whole number.

1. $\sqrt{\frac{1}{5}}$.
2. $\sqrt{\frac{1}{2}}$.
3. $\sqrt{\frac{1}{3}}$.
4. $\sqrt{\frac{2}{3}}$.
5. $-\sqrt{\frac{5}{7}}$.
6. $\sqrt{\frac{1}{n}}$.
7. $4\sqrt{\frac{1}{8}}$.
8. $6\sqrt{\frac{3}{8}}$.
9. $2\sqrt{\frac{2}{3}}$.
10. $5\sqrt{\frac{1}{4}}$.
11. $\frac{1}{2}\sqrt{\frac{3}{5}}$.
12. $-2\sqrt{\frac{5}{3}}$.
13. $\sqrt{\frac{a^2}{6}}$.
14. $\sqrt{\frac{3a}{10}}$.
15. $\sqrt{\frac{2x}{5y}}$.
16. $\sqrt{\frac{6a^2x^4}{5}}$.
17. $\sqrt{1\frac{2}{3}}$.
18. $6\sqrt{1\frac{5}{9}}$.

19. $-\sqrt{3\tfrac{2}{5}}$.

20. $\sqrt{\dfrac{2}{a}}$.

21. $\sqrt{\dfrac{3}{b^3}}$.

22. $\sqrt{\dfrac{5\,a^3}{6\,b^3}}$.

23. $\sqrt{\dfrac{1}{3\,a^2b}}$.

24. $\sqrt{\dfrac{1}{a}+a}$.

Hint: Add first.

25. $2\sqrt[3]{-16}$.

26. $\sqrt[3]{\frac{1}{5}}$.

27. $\sqrt{1+(\frac{1}{3})^2}$.

28. $\sqrt{2-(\frac{1}{2})^2}$.

29. $\sqrt{3a^2+\left(\dfrac{a}{2}\right)^2}$.

30. $\sqrt{a^2+\dfrac{a^2}{4}}$.

129. Transformations Combined with Fundamental Operations. — Radicals which do not appear to be similar can often be made so by transformation. Then they may be added or subtracted.

Illustrative examples:

$2\sqrt{12}+5\sqrt{27}-\sqrt{48}+6\sqrt{\frac{1}{3}}$, when transformed as explained in sections 127 and 128, becomes

$$2\sqrt{4\cdot 3}+5\sqrt{9\cdot 3}-\sqrt{16\cdot 3}+6\sqrt{\tfrac{1}{3}\cdot\tfrac{3}{3}}$$
$$=4\sqrt{3}+15\sqrt{3}-4\sqrt{3}+2\sqrt{3}=17\sqrt{3}.$$

Exercises

Transform by making the radicands integral and as small as possible, and combine similar terms.

1. $2\sqrt{3}+6\sqrt{27}-5\sqrt{48}-2\sqrt{12}$.
2. $3\sqrt{8}-5\sqrt{50}-6\sqrt{2}+\sqrt{98}$.
3. $2\sqrt{125}+4\sqrt{20}+5\sqrt{45}-\sqrt{80}$.
4. $2\sqrt{4\,a}+7\sqrt{9\,a}-3\sqrt{16\,a}-10\sqrt{25\,a}$.
5. $3\sqrt{5}+\sqrt{20}+\sqrt{\frac{1}{5}}$.
6. $8\sqrt{2\,a^5b}-a\sqrt{50\,a^3b}$.
7. $x\sqrt{y}+\sqrt{x^2y}+2\,x\sqrt{y}$.
8. $2\sqrt{27}-6\sqrt{75}-3\sqrt{\frac{1}{3}}+2\sqrt{3}$.

9. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}} + \sqrt{\frac{8}{25}} + 3\sqrt{32}$.
10. $\sqrt{28\,a^3b} - 3\sqrt{63\,ab^3} + \sqrt{175\,a^3b^3}$.
11. $\sqrt{2\,a} + 3\sqrt{8\,a^3} - \sqrt{98\,a^3} + 2\,a\sqrt{\frac{1}{2}\,a}$.
12. $3\sqrt{20} + 5\sqrt{\frac{1}{5}} - \frac{1}{3}\sqrt{45} - 2\sqrt{80}$.
13. $3\sqrt{200} - 6\sqrt{\frac{1}{50}} + 3\sqrt{\frac{1}{15}} + 5\sqrt{60}$.
14. $\sqrt{\frac{1}{3}} + \frac{1}{2}\sqrt{12} - \sqrt{\frac{2}{27}}$.
15. $2\sqrt[3]{16} + 5\sqrt[3]{54} - \sqrt[3]{250} + \frac{1}{2}\sqrt[3]{128}$.
16. $\sqrt[3]{16\,a^3} + 2\sqrt[3]{54\,a^3} - \sqrt[3]{128\,a^3}$.
17. $\sqrt[3]{2\frac{7}{5}} + \sqrt[3]{\frac{8}{5}} - \sqrt[3]{16\frac{1}{5}}$.

130. Multiplication of Monomial Radicals. — Radicals do not have to be similar to be multiplied, but they must be of the same order. That is, we can multiply $\sqrt{3}$ by $\sqrt{5}$, but we cannot multiply $\sqrt{2}$ by $\sqrt[3]{2}$, unless they are first changed so that they have a common index. Multiplication of monomial radicals is performed by the following

Rule. — *Multiply the coefficients for the new coefficient and multiply the radicands for the new radicand, making the index the same as in the radicals multiplied.* That is,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

Illustrative examples:

$2\sqrt{6} \cdot 5\sqrt{3} = 10\sqrt{18} = 30\sqrt{2}$; $(-4\sqrt[3]{2})(3\sqrt[3]{7}) = -12\sqrt[3]{14}$; $(-a\sqrt{b})(-c\sqrt{d}) = ac\sqrt{bd}$.

Exercises

Perform the multiplications indicated, and transform the results as in sections 127 and 128.

1. $2\sqrt{2} \cdot \sqrt{3}$.
2. $5\sqrt{3} \cdot \sqrt{5}$.
3. $\sqrt{2} \cdot 6\sqrt{7}$.
4. $2\sqrt{5} \cdot 3\sqrt{3}$.
5. $2\sqrt{5} \cdot 8\sqrt{2}$.
6. $2\sqrt{6} \cdot 3\sqrt{2}$.

7. $5\sqrt{2} \cdot 6\sqrt{3}$.
8. $2\sqrt{5} \cdot 10\sqrt{6}$.
9. $(-4\sqrt{3})(6\sqrt{2})$.
10. $(8\sqrt{3})(-10\sqrt{2})$.
11. $(-10\sqrt{5})(-4\sqrt{3})$.
12. $(-3\sqrt{7})(-2\sqrt{7})$.
13. $6\sqrt{a} \cdot 10\sqrt{a}$.
14. $(-5\sqrt{b})(3\sqrt{2b})$.
15. $\sqrt{\frac{a}{b}} \cdot \sqrt{\frac{b}{c}}$.
16. $3\sqrt{6} \cdot 5\sqrt{6}$.
17. $(-5\sqrt{3})(2\sqrt{3})$.
18. $(-10\sqrt{7})(-4\sqrt{7})$.
19. $a\sqrt{b^2} \cdot b\sqrt{b}$.
20. $(2\sqrt{2})^2$.
21. $(3\sqrt{5})^2$.
22. $(4\sqrt{3})^2$.
23. $(5\sqrt{6})^2$.
24. $(2\sqrt{5})^2$.
25. $(3\sqrt{3})^2$.
26. $2\sqrt[3]{6} \cdot (-5\sqrt[3]{2})$.
27. $3\sqrt[3]{2} \cdot 10\sqrt[3]{2}$.
28. $a\sqrt[3]{x} \cdot b\sqrt[3]{y}$.
29. $2\sqrt[3]{2} \cdot 5\sqrt[3]{4}$.
30. $(-5\sqrt[3]{9})(-6\sqrt[3]{3})$.

Notice that $(\sqrt{3})^2 = 3$. This follows from the definition of square root. It is a waste of time to say "$(\sqrt{3})^2 = \sqrt{9} = 3$." Similarly $(\sqrt[3]{5})^3 = 5$.

Selection Test

Select the quantity in the parentheses which correctly supplies the answer to each of the following examples:

1. $a^2 \cdot a^5 = (a^7, a^{10})$.
2. $3^2 \cdot 3^5 = (3^7, 3^{10}, 9^7, 9^{10})$.
3. $(x^2)^4 = (x^8, x^6, x^{16})$.
4. $\frac{a^{10}}{a^5} = (a^5, a^2)$.
5. $\frac{5^{10}}{5^5} = (1^5, 1^2, 5^2, 5^5)$.
6. $\sqrt{a^{16}} = (a^4, a^8)$.
7. $\sqrt{a} \cdot \sqrt{b} = (\sqrt{ab}, \sqrt[4]{ab})$.
8. $\sqrt{18} = (2\sqrt{3}, 3\sqrt{2}, 4\sqrt{2})$.
9. $-(-3)^2 = (-9, +9)$.
10. $(\sqrt{x})^6 = (x^2\sqrt{x}, x^3, x^4, x^6)$.

131. Transformation of a Radical with a Coefficient Other than 1 so That Its Coefficient Shall Be 1. — Suppose we wish to change $3\sqrt{5}$ so that it shall have a coefficient of 1. That

is, it must all be under the radical sign and must become what we call an *entire surd*. We know that 3 is the same as $\sqrt{9}$.

Hence $3\sqrt{5} = \sqrt{9} \cdot \sqrt{5} = \sqrt{45}$.

Similarly $\frac{1}{2}\sqrt{6} = \sqrt{\frac{1}{4}} \cdot \sqrt{6} = \sqrt{\frac{6}{4}} = \sqrt{\frac{3}{2}}$.

And $2a\sqrt[3]{2} = \sqrt[3]{8a^3} \cdot \sqrt[3]{2} = \sqrt[3]{16a^3}$.

These illustrations lead to the following

Rule. — *Write the coefficient as a radical whose order is the same as that of the radicand.*

Multiply the radicands.

Illustrative example:

Express $3a\sqrt{5a}$ with coefficient 1.

Solution: $3a$ may be written $\sqrt{9a^2}$.

Hence $3a\sqrt{5a} = \sqrt{9a^2} \cdot \sqrt{5a} = \sqrt{45a^3}$.

Exercises

Express each of the following radicals with coefficient 1.

1. $2\sqrt{5}$.	**6.** $a\sqrt{2}$	**11.** $4\sqrt[3]{2}$.	**16.** $\frac{1}{3}\sqrt{45}$.
2. $3\sqrt{2}$.	**7.** $2a\sqrt{a}$.	**12.** $2\sqrt[3]{-6}$.	**17.** $\frac{2}{3}\sqrt{27}$.
3. $4\sqrt{2}$.	**8.** $3\sqrt{2x}$.	**13.** $ax\sqrt{ax}$.	**18.** $\frac{1}{4}\sqrt{20}$.
4. $5\sqrt{6}$.	**9.** $2\sqrt[3]{6}$.	**14.** $x^2\sqrt{x}$.	**19.** $\frac{1}{2}\sqrt[3]{6}$.
5. $3\sqrt{3}$.	**10.** $3\sqrt[3]{2}$.	**15.** $\frac{1}{2}\sqrt{6}$.	**20.** $\frac{1}{3}\sqrt{12}$.

132. Transformation of a Radical by Reducing Its Order.— Suppose we wish to find the value of $\sqrt[4]{16}$. We do this by taking the square root twice. Thus $\sqrt{16} = 4$, and $\sqrt{4} = 2$. Hence $\sqrt[4]{16} = 2$. Similarly we might proceed if we are asked to transform $\sqrt[4]{4}$. We first take the square root of 4, which is 2. Then when we attempt to take the square root the second time we can only indicate it thus: $\sqrt{2}$. Hence $\sqrt[4]{4} = \sqrt{2}$. Similarly we obtain a sixth root by extracting a cube root and a square root. Thus $\sqrt[6]{27} = \sqrt{3}$

for we may extract the cube root of 27, obtaining 3, but the square root must be indicated.

Exercises

Transform the following radicals so as to reduce their order.

1. $\sqrt[4]{25}$.	**4.** $\sqrt[4]{121}$.	**7.** $\sqrt[6]{25}$.	**10.** $\sqrt[6]{27}$.
2. $\sqrt[4]{49}$.	**5.** $\sqrt[6]{9}$.	**8.** $\sqrt[6]{49}$.	**11.** $\sqrt[6]{125}$.
3. $\sqrt[4]{100}$.	**6.** $\sqrt[6]{4}$.	**9.** $\sqrt[6]{8}$.	**12.** $\sqrt[6]{216}$.

133. Multiplication of Binomial Surds. — A binomial at least one of whose terms is a radical is called a *binomial surd.* Examples are $3+\sqrt{6}$, $5-2\sqrt{3}$, $2\sqrt{3}+5\sqrt{2}$, and $a-\sqrt{b}$. The method of multiplying a monomial surd by a binomial surd or of multiplying two binomial surds is illustrated by the following examples.

Illustrative example I:

Multiply $3\sqrt{6}+2\sqrt{3}$ by $3\sqrt{2}$.

Solution. $3\sqrt{2}(3\sqrt{6}+2\sqrt{3}) = 9\sqrt{12}+6\sqrt{6}$

$= 9\sqrt{4\cdot 3}+6\sqrt{6}$

$= 18\sqrt{3}+6\sqrt{6}$.

Illustrative example II:

Multiply $2\sqrt{3}+\sqrt{6}$ by $3\sqrt{3}-2\sqrt{6}$.

Solution:

$$\begin{array}{rl} 2\sqrt{3}+\sqrt{6} & \\ 3\sqrt{3}-2\sqrt{6} & \\ \hline 18+9\sqrt{2} & \text{Product of } (2\sqrt{3}+\sqrt{6}) \text{ by } 3\sqrt{3}. \\ -12-12\sqrt{2} & \text{Product of } (2\sqrt{3}+\sqrt{6}) \text{ by } -2\sqrt{6}. \\ \hline 6-3\sqrt{2} & \text{Sum of partial products.} \end{array}$$

You should note that each term in the product is transformed as soon as it is obtained.

For example

$$2\sqrt{3} \cdot 3\sqrt{3} = 6\sqrt{9} = 6 \cdot 3 = 18.$$

Similarly

$$\sqrt{6} \cdot 3\sqrt{3} = 3\sqrt{18} = 3\sqrt{9 \cdot 2} = 3 \cdot 3\sqrt{2} = 9\sqrt{2}.$$

Exercises

Perform each of the indicated multiplications, transforming each result so as to make the radicand as small as possible.

1. $\sqrt{3}(2\sqrt{2} - \sqrt{3})$.
2. $3\sqrt{2}(4\sqrt{2} - 1)$.
3. $5\sqrt{3}(2\sqrt{3} - 3\sqrt{2})$.
4. $\sqrt{5}(5\sqrt{3} + 3\sqrt{5})$.
5. $2\sqrt{6}(\sqrt{3} - 2\sqrt{6} + \sqrt{12})$.
6. $-\sqrt{3}(2\sqrt{6} + 3\sqrt{3} - 5\sqrt{18})$.
7. $3\sqrt{a}(\sqrt{a} + \sqrt{b})$.
8. $-2\sqrt{ab}(\sqrt{a} - \sqrt{b})$.
9. $(3 + \sqrt{2})(3 - \sqrt{2})$.
10. $(3\sqrt{2} - \sqrt{3})(3\sqrt{2} + \sqrt{3})$.
11. $(5\sqrt{6} + \sqrt{5})(5\sqrt{6} - \sqrt{5})$.
12. $(2 + 3\sqrt{5})(2 - 3\sqrt{5})$.
13. $(\sqrt{3} - 2\sqrt{6})(4\sqrt{3} + \sqrt{6})$.
14. $(5\sqrt{2} + 6)(8\sqrt{2} - 1)$.
15. $(4\sqrt{2} + 3\sqrt{5})(2\sqrt{2} - 4\sqrt{5})$.
16. $(2 - 2\sqrt{x})^2$.
17. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.
18. $(2\sqrt{a} + b)(a - 2\sqrt{b})$.
19. $(3\sqrt{x} + 5\sqrt{y})(2\sqrt{x} - 3\sqrt{y})$.
20. $(2 - 3\sqrt{x})^2$.

134. Division of Radicals. — Only radicals of the same order can be divided. The operation is performed by the following

Rule. — *Divide the coefficient of the dividend by that of the divisor to obtain the coefficient of the quotient, and divide the radicand of the dividend by that of the divisor to obtain the radicand of the quotient.* This must be of the same order as the dividend and divisor. That is,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Illustrative examples:

$$\frac{6\sqrt{6}}{2\sqrt{3}} = 3\sqrt{2}. \quad \frac{-15\sqrt[3]{10}}{3\sqrt[3]{2}} = -5\sqrt[3]{5}. \quad \frac{3\sqrt{18}}{-2\sqrt{3}} = -\frac{3}{2}\sqrt{6}.$$

$$\frac{3\sqrt{3}}{5\sqrt{6}} = \frac{3}{5}\sqrt{\frac{1}{2}} = \frac{3}{5}\sqrt{\frac{1}{2}\cdot\frac{2}{2}} = \frac{3}{5}\sqrt{\frac{2}{4}} = \frac{3}{5}\cdot\frac{1}{2}\sqrt{2} = \frac{3}{10}\sqrt{2}.$$

Oral Exercises

Divide:

1. $\dfrac{\sqrt{12}}{\sqrt{2}}$.
2. $\dfrac{2\sqrt{6}}{\sqrt{3}}$.
3. $\dfrac{-3\sqrt{18}}{\sqrt{2}}$.
4. $\dfrac{-12\sqrt{24}}{2\sqrt{8}}$.
5. $\dfrac{10\sqrt{40}}{-2\sqrt{5}}$.
6. $\dfrac{16\sqrt{70}}{8\sqrt{5}}$.
7. $\dfrac{-12\sqrt{48}}{-4\sqrt{6}}$.
8. $\dfrac{10\sqrt[3]{81}}{2\sqrt[3]{9}}$.
9. $\dfrac{2\sqrt[4]{128}}{3\sqrt[4]{32}}$.

Written Exercises

Divide:

1. $\dfrac{\sqrt{3}}{2\sqrt{6}}$.
2. $\dfrac{4\sqrt{2}}{2\sqrt{3}}$.
3. $\dfrac{5\sqrt{15}}{2\sqrt{5}}$.

4. $\dfrac{-6\sqrt{3}}{\sqrt{6}}$.

5. $\dfrac{3\sqrt{5}}{-\sqrt{10}}$.

6. $\dfrac{4\sqrt{2}}{3\sqrt{8}}$.

7. $\dfrac{7\sqrt{2}}{3\sqrt{3}}$.

8. $\dfrac{5\sqrt{12}}{3\sqrt{6}}$.

9. $\dfrac{8\sqrt{15}}{-4\sqrt{5}}$.

Divide:

10. $5\sqrt{30}$ by $8\sqrt{6}$.
11. $6\sqrt{2}$ by $2\sqrt{3}$.
12. $11\sqrt{12}$ by $2\sqrt{5}$.
13. 4 by $3\sqrt{6}$.
14. $3\sqrt{15}$ by $2\sqrt{60}$.
15. $3\sqrt{10}$ by $2\sqrt{2}$.

Find the value of:

16. $14 \div (-2\sqrt{7})$.
17. $-30 \div (-2\sqrt{15})$.
18. $3\sqrt[3]{12} \div 2\sqrt[3]{3}$.
19. $5\sqrt[3]{18} \div 3\sqrt[3]{2}$.
20. $2\sqrt[3]{2} \div 3\sqrt[3]{6}$.

135. Principal Root. — You will note that in working with radicals we are not accustomed to consider the square root of a number as being both + and −. Instead we consider only the positive square root, which is called the *principal root.*

Thus 2 is the principal root of $\sqrt{4}$; 3 is the principal root of $\sqrt{9}$; $2\sqrt{9} - 3\sqrt{25} - 2\sqrt{49} = 2 \cdot 3 - 3 \cdot 5 - 2 \cdot 7 = 6 - 15 - 14 = -23$, etc.

136. Simplification of Radicals. — A radical is often said to be *simplified* when it is transformed in any of the following ways: (**a**) so as to make the order of the radical as low as possible, (**b**) so as to make the radicand as small as possible and a whole number, (**c**) so there shall not be a radical in the denominator of a fraction. For many purposes, however, the radical is not really any simpler because of these changes.

Exercises

"Simplify" each of the following radicals:

1. $3\sqrt{24}$.
2. $-5\sqrt{60}$.
3. $\frac{a}{b}\sqrt{3\,b^2}$.
4. $\sqrt{120\,a^2b^3}$.
5. $\sqrt{\frac{3}{8}}$.
6. $2\sqrt{\frac{1}{2}}$.
7. $3\sqrt{\frac{2}{3}}$.
8. $\sqrt{\frac{x^2}{b}}$.
9. $\sqrt{2\frac{1}{2}}$.
10. $\sqrt{3\frac{2}{3}}$.
11. $\sqrt{9}$.
12. $\sqrt{25}$.
13. $\sqrt{100}$.
14. $\sqrt[3]{125}$.
15. $\frac{3\sqrt{6}}{2\sqrt{2}}$.
16. $\frac{12\sqrt{5}}{3\sqrt{2}}$.
17. $\frac{2}{\sqrt{5}}$.
18. $\frac{3}{\sqrt{2}}$.
19. $\frac{1}{\sqrt{3}}$.
20. $\frac{3\sqrt{6}}{2\sqrt{5}}$.
21. $\sqrt[4]{9}$
22. $\sqrt[6]{25}$

137. The Fractional Exponent. — It has already been explained that $\sqrt{x^6} = x^3$, and $\sqrt[3]{x^{12}} = x^4$; that is, in the extraction of roots, the exponent of the power is divided by the index of the root to obtain the exponent in the result.

The application of this rule may result in a fractional exponent, thus, $\sqrt[3]{x^2} = x^{\frac{2}{3}}$. The fractional exponent may, therefore, be used instead of the radical sign to indicate an irrational number. When it is so used, you should note that the numerator of the fraction indicates the power to be taken and the denominator indicates the root to be extracted.

Illustrative examples:

$$x^{\frac{1}{2}} = \sqrt{x}; \quad a^{\frac{3}{2}} = \sqrt{a^3} = \sqrt{a^2 \cdot a} = a\sqrt{a}.$$

Exercises

Write with fractional exponents:

1. $\sqrt{3}$.
2. $\sqrt[3]{x}$.
3. $\sqrt[3]{a^2}$.
4. $\sqrt[3]{ax}$.
5. $\sqrt[4]{x^3}$.
6. $\sqrt{m^3}$.
7. $\sqrt{x}$.
8. $\sqrt[5]{x^4}$.

Write with radical signs and simplify when possible:

9. $a^{\frac{2}{3}}$.
10. $3^{\frac{3}{4}}$.
11. $2^{\frac{1}{2}}$.
12. $x^{\frac{3}{2}}$.
13. $(ay)^{\frac{4}{5}}$.
14. $3\,x^{\frac{2}{3}}$.

Simplify:

15. $(16\,m)^{\frac{1}{4}}$.
16. $18^{\frac{1}{2}}$.
17. $27^{\frac{1}{3}}$.
18. $16^{\frac{1}{3}}$.
19. $m^{\frac{3}{5}}$.
20. $(ax)^{\frac{5}{4}}$.
21. $32^{\frac{2}{5}}$.
22. $64^{\frac{5}{6}}$.
23. $32^{\frac{1}{5}}$.
24. $(-32)^{\frac{1}{5}}$.
25. $8^{\frac{1}{3}} \cdot 16^{\frac{1}{4}}$.
26. $\dfrac{16^{\frac{3}{4}} \cdot 32^{\frac{1}{5}} \cdot 8^{\frac{2}{3}}}{8^{\frac{1}{3}} \cdot 64^{\frac{2}{3}} \cdot 16^{\frac{1}{2}}}$.

138. Imaginary Numbers. — Since the square of either a positive number or a negative number is always positive, it follows that no real number, when squared, can produce a negative number. The square root of a negative number is therefore neither positive nor negative. Such a number is called *imaginary*, and will receive no further consideration in this book. $\sqrt{-2}$ and $2\sqrt{-9}$ are illustrations of imaginary numbers.

You should note, however, that the cube root and all odd roots of a negative number are not imaginary, but negative.

Thus, $\sqrt[3]{-8} = -2$.

Review

1. If $a = 6$ and $b = 1$, what does $2\,ab - ab^2$ equal?
2. If $a = 6$ and $b = 3$, what does $\sqrt{a}\sqrt{2\,ab}$ equal?
3. How do you find the value of x in $x + 14 = 2\,x + 8$?
4. If $\dfrac{1}{a} - \dfrac{1}{x} = \dfrac{1}{x} - \dfrac{1}{5}$, what does x equal?
5. Find the square root of 33.1776.

6. Find the value of $\dfrac{8x + 5y - 6z}{2a + 3b}$ if $\begin{matrix} x = 1, & a = 4, \\ y = 2, & b = 5. \\ z = 3, & \end{matrix}$

7. If 3 feet be added to both sides of a certain square, its area is increased by 81 square feet. Find the side of the original square.

8. Find the value of y in

$$.15y + \frac{.135y - .225}{.6} = \frac{.36}{.2} - \frac{.09y - .18}{.9}.$$

9. Simplify $\sqrt{a^3 + \dfrac{a^3}{8}}$.

10. How much water must be added to a quart of 95% pure alcohol to make a solution 40% pure?

11. Show that

$$(1 - a)(1 - b)(1 - c) + a(1 - b)(1 - c) + b(1 - c) + c = 1.$$

12. Find the numerical value of

$$(a - b)^2 + (b - c)^2 + (c - a)^2 + (b - c)(c - a)(a - b)$$

when $a = 1$, $b = 2$, and $c = \frac{1}{2}$.

13. Simplify: $\sqrt{(x + y)^3} - \sqrt{(x^2 - y^2)(x - y)}$.

14. Solve $C = \dfrac{E}{R + r}$ for E; for R; for r.

How does C change when E is increased if R and r remain the same? If E and r remain the same and R is increased?

15. (a) What is the sign of an odd power of a negative quantity? (b) Of an even power? (c) Of all powers of positive quantities?

16. Solve the equation $\dfrac{x}{10} - \dfrac{2x}{5} = \dfrac{3}{2}$.

17. Will $x = \sqrt{10} + \sqrt{5}$ and $y = \sqrt{10} - \sqrt{5}$ check in the equation $x^2 - 3xy - 10\sqrt{2} = 0$?

18. Factor $10x^2 - 9x - 7$.

19. If $3a^3 - 11a^2 + 9a + m$ is exactly divisible by $3a - 2$, what is the value of m?

Exercises

Problems Involving Square Root and Radicals

1. Find the side of a square whose area is 27 square inches.

2. What is the radius of a circle whose area is 24.2 square feet?

Hint: The area of a circle equals πr^2, where $\pi = 3.1416$ and r represents the radius of the circle.

3. What is the diagonal of a square whose side is 6?

4. What is the diagonal (d) of a square whose side is (s)?

5. By means of the equation obtained in number 4, what is the diagonal of a square whose side is 8? 10? 15? 20?

6. Solve your answer obtained in example 4 for the value of s.

7. What is the side of a square whose diagonal is $\sqrt{50}$?

8. Draw a square (preferably on squared paper) 4 units (or quarter inches) on a side. Measure the diagonal, then find the diagonal by the use of example 4. Compare your results. Which method is simpler?

9. (a) What is an equilateral triangle? Draw one.

(b) If one side of an equilateral triangle is 6, find the altitude a.

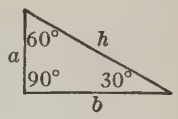
10. Find the side (s) of an equilateral triangle whose altitude is a.

11. What is the side of an equilateral triangle whose altitude is 27?

12. If a horse is tied to a stake by a tether 30 feet long, over how many square feet may he graze?

Hint: See example 2 above.

13. The hypotenuse of a right triangle is 10. Each leg of the triangle is x. Find the value of x. Find the area of the triangle.



14. The hypotenuse of a right triangle, one of whose angles is 30° and the other 60° (called a 30°–60° triangle), is twice the length of the leg opposite the 30° angle. If this leg is $5\sqrt{2}$, what is the hypotenuse? What is the length of the other leg?

Ladder against a Burning House

15. Find the number of square inches in the surface of a piece of sheet iron in the shape of a 30°–60° triangle if the hypotenuse is $8\sqrt{2}$.

16. If $s = 4\pi R^2$, where s = area of the surface of a sphere and R = radius of the sphere, what is the area of a sphere whose radius is 10 inches?

17. The foot of a ladder is placed 10 feet from the side of a house and it will just reach a window 40 feet from the ground. How long is the ladder?

18. The legs of a right triangle are 27.68 feet and 34.35 feet. Find the hypotenuse.

139. Formulas Containing Radicals. — Formulas often contain radicals when solved for certain letters. In the case of formulas only the positive value of the radical should be used.

Exercises

Answer Questions similar to those on page 268.

1. (a) Solve for V, $K = \frac{1}{2}mV^2$.

(b) Compute V to the nearest hundredth if $K = 40$ and $m = 6$. Three decimals necessary.

(c) if $K = 40$ and $m = 12$.

(d) if $K = 20$ and $m = 6$.

(e) if $K = 80$ and $m = 12$.

2. (a) Solve for t, $S = \frac{1}{2}gt^2$.

(b) Compute to the nearest tenth the value of t if $S = 150$ and $g = 32$. Two decimal places required.

(c) if $S = 50$ and $g = 32$.

(d) if $S = 300$ and $g = 32$.

3. (a) Solve for r, $A = \pi r^2$.

(b) Compute r if $A = 154$ and $\pi = \frac{22}{7}$.

(c) if $A = 77$ and $\pi = \frac{22}{7}$.

(d) if $A = 308$ and $\pi = \frac{22}{7}$.

4. (a) Solve for r, $S = 4\pi r^2$.

(b) Find r to the nearest tenth if $S = 242$.

(c) if $S = 121$. (d) $S = 484$.

5. Solve for r, $V = \frac{4}{3}\pi r^3$.

6. (a) Solve for r, $V = \pi r^2 h$.

(b) Compute the value of r to the nearest hundredth if $V = 160$, and $h = 8$.

(c) if $V = 160$ and $h = 4$.

(d) if $V = 160$ and $h = 16$.

(e) if $V = 320$ and $h = 8$.

7. (a) Solve for b, $a^2 + b^2 = c^2$.

(b) Find b if $c = 17$ and $a = 8$.

(c) if $c = 34$ and $a = 16$.

(d) if $c = 51$ and $a = 24$.

8. (a) Solve for b, $\frac{a}{b} = \frac{b}{c}$.

(b) Compute b if $a = 4$ and $c = 9$.

(c) Compute b to the nearest hundredth if $a = 6$ and $c = 12$. (d) $a = 2, c = 9$. (e) $a = 4, c = 18$. (f) $a = 8, c = 18$.

9. (a) Solve for s, $d = s\sqrt{2}$.

(b) Find s (the side of a square) when the diagonal $(d) = 8.484$. (c) if $d = 4.242$. (d) if $d = 16.968$.

10. (a) (See footnote on next page.) $\text{Area}_{\square} = b\sqrt{d^2 - b^2}$, where d = diagonal of the rectangle and b = the base of the rectangle. Find the area if $b = 3$ and $d = 6.7083 = 3\sqrt{5}$.

(b) if $b = 3$ and $d = 6\sqrt{5}$.

(c) if $b = 6$ and $d = 3\sqrt{5}$.

(d) if $b = 6$ and $d = 6\sqrt{5}$.

11. (a) $\text{Area}_{\triangle} = \sqrt{s(s-a)(s-b)(s-c)}$,

where a = one side of triangle,

b = another side of triangle,

c = third side of triangle,

$s = \frac{a+b+c}{2}$.

Illustrative example:

Find the area of a triangle whose sides are 15, 16, and 17.

Solution:

$$A_{\triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{24 \cdot 9 \cdot 8 \cdot 7}$$
$$= \sqrt{2^3 \cdot 3 \cdot 3^2 \cdot 2^3 \cdot 7}$$
$$= \sqrt{2^6 \cdot 3^2 \cdot 3 \cdot 7}$$
$$= 2^3 \cdot 3\sqrt{21}$$
$$= 24\sqrt{21}$$
$$= 24(4.582)$$
$$= 109.968. \quad \textit{Answer.}$$

$a = 15$,

$b = 16$,

$c = 17$,

$s = \frac{15+16+17}{2} = 24$,

$s - a = 9$,

$s - b = 8$,

$s - c = 7$.

(b) Find the area of a triangle whose sides are 6, 8, and 12.

(c) 6, 8, and 6. (d) 3, 4, and 6.

(e) 12, 16, and 24. (f) each side 10.

12. Using the equation $h = \frac{gt^2}{2}$, where

h = height in feet from which an object falls,
g = 32.16 feet per second each second,
t = time in seconds,

find how long it will take an object to fall to the earth from a height of 700 feet.

Boston Light

13. A body B is h feet above the surface of the earth. The number of miles (m) at which it may be seen is denoted by the equation $m = \sqrt{\frac{3\,h}{2}}$. If the light in Boston light-

In formulas the expressions "area of a rectangle" and "area of a triangle" are abbreviated "Area$_{\square}$" and "Area$_{\triangle}$" or simply "$A_{\square}$" and "$A_{\triangle}$." See examples 10 and 11.

house is 200 feet above the sea level, what is the greatest distance from which it may be seen?

If it were 400 feet above sea level? 100 feet above sea level?

14. $C_o = 2\sqrt{\pi A}$ where C_o = circumference of a circle and A = area of the circle. Find C if $A = 36$; $A = 18$; $A = 72$.

15. (a) Solve $\pi d^2 = 4A$ for d.

(b) Find d when $A = 176$ and $\pi = \frac{22}{7}$.

(c) $A = \frac{88}{7}$ and $\pi = \frac{22}{7}$.

(d) $A = \frac{352}{7}$ and $\pi = \frac{22}{7}$.

True-False Exercise

Copy the numbers of the following examples and place a t after each of those that you believe correct and an f after each one that you consider incorrect:

1. In the formula $C = \pi d$, if $d = 28$ and $\pi = \frac{22}{7}$, then $C = 88$.

2. In the formula $A = bh$, if $b = 12$ and $h = 6$, then $A = 18$.

3. In the formula $t = \frac{d}{r}$, if $d = 120$ and $r = 30$, then $t = 3$.

4. In the formula $F = 1.8\,C + 32$, if $C = 10°$, then $F = 40°$.

5. In the formula $S = 2\pi rh$, if $\pi = \frac{22}{7}$, $r = 7$ and $h = 4$, then $S = 166$.

6. The formula for the area of a square is $A = s^2$.

7. The formula for the volume of a cube is $V = e^3$.

8. The formula for the area of a triangle is $A = bh$.

9. The formula for the area of a rectangle is $A = \frac{bh}{2}$.

10. The formula for the total surface of a cube is $S = 6e^2$.

Review Exercises

1. What must be added to $3x + 4y - 6z^2$ so that the sum may be $2x - 3y + 6z^2$?

2. Find a number which when multiplied by 3 exceeds 16 by as much as 16 exceeds the original number.

3. Factor and simplify:

$$\frac{a^2 - 11a + 30}{a^3 - 6a^2 + 9a} \times \frac{a^2 - 3a}{a^2 - 25} \div \frac{a^2 - 9}{a^2 + 2a - 15}.$$

Check by substituting 2 for a.

4. Simplify $2\sqrt{150} - 4\sqrt{54} + 6\sqrt{24}$.

5. The figure represents a steel plate which had the form of a square of side a, from which four smaller squares, each of side b, have been cut.

(a) Find the area of the plate as shown in the figure.

(b) If $a = 96$ and $b = 2$, what is the area? Explain a short process for finding the latter area.

6. The dimensions of a rectangular block are e, $e + 3$, and $e + 5$.

(a) Find (1) the sum of all of the edges;
(2) the total surface of the block;
(3) the volume of the block.

(b) If $e = 3$, find the value of each of these expressions.

7. Solve $4x + 5 = 2x - 7$.

State what operation is used in each step of your work.

8. Find the square root of 8114.4064. What is the square root of .0081144064? of 811440.64? Can you determine the square root of .081144064 from any of the above?

9. How much water must be added to 80 pounds of a 5% salt solution to obtain a 4% salt solution?

10. Reduce to an improper fraction:

$$a^2 - ab + b^2 - \frac{b^3}{a+b}.$$

11. Factor each of the following:

(a) $4x^2 + 11x - 3$.
(b) $x^3 - 7x^2 + 6x$.
(c) $(a+b)^2 - (a-b)^2$.
(d) $a^2 - a - 12$.
(e) $6a^2 + 7a + 2$.
(f) $8 - 18m^2$.

12. Solve $\frac{1}{2}(x+1) - \frac{x}{3}(2x-1) = \frac{2}{3}$. Check the result.

13. Find the value of $ab^3 - 3cd^2 + 2(3a - b)(c - 2d)$ when $a = 4$, $b = -3$, $c = 2$, $d = -4$.

14. (a) Express in dollars p per cent of d dollars.

(b) Express in square yards the area of a rectangle that is a feet long and 9 feet wide.

(c) State what value of x will make the expression $3(x+2) - 4(x-3)$ equal to twice the value of x.

15. Simplify $2\sqrt{54} - 6\sqrt{\frac{2}{3}} - \sqrt{96}$.

16. Write in words the arithmetic rule for dividing one fraction by another fraction. Express the rule in algebraic symbols if $\frac{a}{b}$ is the dividend and $\frac{c}{d}$ is the divisor.

17. What is the coefficient of z in $5xyz$? Of y in $\frac{1}{2}xy$? Of a in the expression $a(b-c)(c-d)$?

18. Write in words $\frac{(a+b)^2}{c^3}$.

19. Multiply $\frac{1}{2}y^2 - by + \frac{1}{2}b^2$ by $\frac{2}{3}by + \frac{1}{2}y^2$.

20. Divide $3 - y$ by $1 + y$ to three terms. State what the remainder is.

21. What number must be added to both the numerator and the denominator of $\frac{11}{12}$ to equal the fraction $\frac{2}{3}$?

22. Write a trinomial which may be factored into a monomial and the square of a binomial.

23. Solve the equation

$$\frac{.5\,x}{.3} + \frac{.7\,x}{.9} - \frac{.8}{1.2} = 7.$$

24. Write all of the powers of 2 from the first to the tenth.

25. $2^2 \cdot 2^3 \cdot 2^4 = 2^? = ?$

26. Complete the following tabulation by supplying the values of $x^2 + x + 1$ when x has the different values indicated. How does $x^2 + x + 1$ change as x increases?

$x =$	$x^2 + x + 1 =$
0	
1	
2	
3	

27. The horse power of a certain gasoline engine is denoted by $H = \frac{1}{6} D^2N$. Find the horse power of such an engine when D (the diameter of a cylinder in inches) = 5, and N (the number of cylinders) = 2.

28. What would be the meaning if you found a negative result for the following:

(a) The amount of money you had made during the year?

(b) The rate at which a deer ran away from a hunter?

(c) The increase in population of a city during the past 10 years?

(d) The height of a piece of land above sea level?

(e) The lifting power of a balloon?

29. When is the minuend in subtraction less than the remainder?

30. Change the following word expressions into the corresponding algebraic expressions:

(a) Three times the length of a side increased by eight.

(b) The perimeter of a square equals four times a side.

(c) The area of a square equals the square of a side.

(d) The area of a certain rectangle equals the product of its length and width, in the number of like units.

(e) The perimeter of a rectangle equals twice the sum of two adjacent sides (b and a).

31. Write in the form of a fraction the following:

Six times the square of x increased by nine times x, increased by three, divided by five times a quantity which exceeds x by three.

32. Divide a line into two parts such that one part shall be 7 inches longer than the other and six times the smaller part shall equal four times the longer part.

33. Solve $\frac{5y}{6} - \frac{1}{2} + \frac{3}{8}\left(y - \frac{1}{3}\right) + \frac{7}{24} = 0$ and check.

34. Find the remainder when the sum of the squares of x and y is subtracted from the square of the sum of x and y.

35. Solve the equation $x = \frac{D}{2} - y$ for D. For y.

36. A man has \$14 in quarters and half dollars. He has five times as many quarters as half dollars. How many has he of each?

37. An airplane can make 95.7 miles in still air. How fast can it go against a wind blowing 8.9 miles per hour? Blowing $(x + y)$ miles an hour?

38. If a man sells an article so as to gain 15% on the cost, find the cost (c) in terms of the selling price (s).

Hint: $s = c + .15\,c$.

39. If $3x - 6$ and $2x + 10$ represent the same number, what is the value of x?

40. Simplify: $\dfrac{3}{x+2} + \dfrac{2}{x+3}$.

True and False Test

State whether each of the following statements is true or false:

1. $(a^4)^2 = a^8$.

2. $a^2 \cdot a^3 = a^6$.

3. The only square root of 9 is 3.

4. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.

5. $\sqrt{2} \cdot \sqrt{3} = \sqrt[4]{6}$.

6. $\dfrac{2^6}{2^2} = 2^4$.

7. $\dfrac{3^4}{3^2} = 1^2$.

8. $\sqrt{8} = 2\sqrt{2}$.

9. $\sqrt{a^9} = a^3$.

10. $\sqrt{x^6} = x^3$.

11. $3^2 \cdot 3^3 = 9^5$.

12. $\sqrt[4]{4} = \sqrt{2}$.

13. $\sqrt{a^2 + b^2} = a + b$.

14. $\sqrt{m^2 + n} = m + \sqrt{n}$.

15. $\sqrt{4} + \sqrt{9} = \sqrt{4 + 9}$.

16. The square root of -9 is -3.

CHAPTER XII

THE QUADRATIC EQUATION

So far our work has been focused on the linear equation, its development through the formula, its solution, and its graphing. When we supplement our knowledge of linear equations with what we learned in the last chapter about powers and roots, we are ready to solve quadratic equations.

140. Kinds of Quadratic Equations. — A *quadratic equation* is one which after simplifying contains the second power (square) of the unknown quantity but no higher power. When a quadratic equation is of the form $ax^2 + bx + c = 0$, it is called a *complete* quadratic equation. This means that it contains three terms, one having the square of the unknown, another having the first power of the unknown, and a third not having the unknown. If either the term containing the first degree of the unknown or the constant term (without the unknown) or both are lacking, it is called an *incomplete* quadratic equation.

Thus $2x^2 - 7x + 6 = 0$ is a complete quadratic equation, while $3x^2 + 5x = 0$, $2x^2 - 3 = 0$, and $5x^2 = 0$ are incomplete quadratic equations.

141. Solving Incomplete Quadratics. — When the constant term is lacking, as in $3x^2 + 5x = 0$, we may solve the equation by factoring as in section 84.

Illustrative example I:

$$3x^2 + 5x = 0.$$

$$x(3x + 5) = 0.$$

$$x = 0. \quad 3x + 5 = 0. \quad 3x = -5. \quad x = -\tfrac{5}{3}.$$

Check by substituting each of the results in the original equation.

Hence $x = 0$, or $-\frac{5}{3}$, roots.

Notice that in this type of example x is always a factor and 0 is always one of the roots of the equation.

When the term containing the first power of the unknown is lacking, as in $2x^2 - 9 = 0$, we should proceed as shown in the following

Illustrative example II:

Solve $2x^2 - 9 = 0$

Solution:

$$2x^2 = 9.$$
$$x^2 = \tfrac{9}{2}.$$
$$x = \pm\sqrt{\tfrac{9}{2}}$$
$$= \pm\sqrt{\tfrac{18}{4}} = \pm\tfrac{3}{2}\sqrt{2}.$$

When both the second and third terms are lacking, as in $3x^2 = 0$, both roots are always 0, and this case need not be further considered.

Exercises

Solve each of the following equations. Check.

1. $x^2 - 12x = 0.$
2. $x^2 + 2x = 0.$
3. $x^2 + 20x = 0.$
4. $x^2 - 3x = 0.$
5. $2x^2 - 5x = 0.$
6. $3x^2 + 2x = 0.$
7. $5x^2 - x = 0.$
8. $mx^2 - nx = 0.$
9. $ax^2 + bx = 0.$
10. $8x^2 + x = 0.$
11. $2x^2 = -7x.$
12. $6x^2 - 3x = 2x^2 + 5x.$
13. $x^2 + 10x = 3x + 5x^2.$
14. $9x^2 - 3ax + 5x^2 = -6ax.$
15. $\dfrac{x^2}{2} + \dfrac{x}{6} = 0.$
16. $\dfrac{3x^2}{5} - \dfrac{2x}{3} = 0.$

17. $x^2 - 16 = 0$.
18. $x^2 = 4$.
19. $4x^2 - 25 = 0$.
20. $4x^2 = 49$.
21. $9x^2 - 4 = 0$.
22. $25x^2 - 1 = 0$.
23. $x^2 - a^2 = 0$.
24. $4x^2 - b^2 = 0$.
25. $16x^2 = 25b^2$.
26. $9x^2 - 49a^2c^2 = 0$.
27. $4x^2 = 81a^2$.
28. $64x^2 - 49a^2c^2 = 0$.
29. $x^2 - 3 = 0$.
30. $x^2 - 6 = -3$.
31. $2x^2 - 15 = 0$.
32. $3x^2 - 5 = 0$.
33. $4x^2 = 3$.
34. $5x^2 - 1 = 0$.
35. $10x^2 = 7$.
36. $4x^2 - 9 = 0$.
37. $x^2 - a = 0$.
38. $ax^2 - c = 0$.
39. $5x^2 = ab$.
40. $7x^2 - 20 = 5x^2 + 78$.
41. $4x^2 - 1 = x^2 + 5$.
42. $-3 - 2x^2 = 1 - 5x^2$.
43. $10x^2 - 3 = 2x^2 + 2$.
44. $\dfrac{x^2 - 4x}{2} = \dfrac{5x^2 - 6x - 14}{3}$.
45. $\dfrac{2x}{2 - x} = \dfrac{2 + x}{2x}$.
46. $\dfrac{x + 1}{2x} + \dfrac{3x - 3}{6x} = 2x$.
47. $\dfrac{5x}{3 - 2x} - \dfrac{3 + 2x}{2x} = 0$.

Problems Which Use Incomplete Quadratics

1. Four times the square of a number decreased by the number equals 0. Find the number.

2. What is the number which if two times its square be added to three times the number the result is 0?

3. Six times the square of a number equals seven times the number. What is it?

4. The square of a number divided by 2 added to the result of five times the number divided by 3 equals 0. What is the number?

5. Find the number whose square equals 25.

6. Two times the square of a number equals 72. What is the number?

7. Two times the square of a number equals 7. Find the number.

8. The area of a rectangular field is 125 square rods. The field is 5 times as long as it is wide. Find the dimensions.

9. The base of a triangle is 7 times the altitude. The area is 126. Find the base and the altitude.

142. Solving Complete Quadratics by Completing the Square. — The simplest way to solve quadratics is by factoring, but not all quadratics can be factored. We shall now study a method by means of which *any* quadratic equation can be solved. It is called *completing the square.*

Illustrative example I:

Solve $x^2 + 6x = 16$.

Solution: If 9 is added to the left member, the left member then becomes a perfect square. If it is added to the left member, it must be added to the right member also. We get the 9 by taking one half of 6 (the coefficient of x) and squaring it. We then obtain

$$x^2 + 6x + 9 = 25.$$

$x + 3 = \pm 5$. By extracting the root of both members.

$x + 3 = 5$	$x + 3 = -5$
$x = 2$, *root*	$x = -8$, *root*
$\therefore x = 2$ or -8, *roots*	
Check: $x = 2$	*Check:* $x = -8$
$4+12+9 = 25$	$64-48+9 = 25$
$25 = 25$	$25 = 25$

This illustrates the following

Rule. — (1) *Remove parentheses, clear the equation of fractions, and combine similar terms.*

(2) *Reduce the equation to the form* $x^2 + px = q$.

(3) *Complete the square by adding to both members of the equation the square of one half of the coefficient of* x.

(4) *Extract the square root of both members of the equation, prefixing the* $\pm$ *sign to the right member.*

(5) *Solve the resulting equations.*

(6) *Check each root in the original equation.*

Illustrative example II:

Solve the equation $2x^2 + 3x - 5 = 0$. Reduce the equation to the form $x^2 + px = q$ by adding $+5$ to both members and dividing by 2.

Solution:

$$2x^2 + 3x = 5.$$

$$x^2 + \frac{3}{2}x = \frac{5}{2}.$$ Dividing by 2 (the coefficient of x^2).

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{49}{16}.$$ Part (3) of Rule.

$$x + \frac{3}{4} = \pm\frac{7}{4}.$$ Extracting root.

$x + \frac{3}{4} = \frac{7}{4}$	$x + \frac{3}{4} = -\frac{7}{4}$
$x = 1$, *root*	$x = -\frac{5}{2}$, *root*
$x = 1$ or $-\frac{5}{2}$, *roots*	
Check: $x = 1$	*Check:* $x = -\frac{5}{2}$
$2 + 3 = 5$	$\frac{25}{2} - \frac{15}{2} = 5$
$5 = 5$	$5 = 5$

Illustrative example III:

Solve $x^2 - 6x - 2 = 0$.

$$x^2 - 6x = 2$$
$$x^2 - 6x + 9 = 11$$
$$(x - 3)^2 = 11$$
$$x - 3 = \pm\sqrt{11}$$

$x - 3 = +\sqrt{11}$	$x - 3 = -\sqrt{11}$
$x = 3 + \sqrt{11}$	$x = 3 - \sqrt{11}$

Check: $x = 3 + \sqrt{11}$

$$(3 + \sqrt{11})^2 - 6(3 + \sqrt{11}) - 2 = 0$$
$$9 + 6\sqrt{11} + 11 - 18 - 6\sqrt{11} - 2 = 0$$
$$0 = 0$$

Check: $x = 3 - \sqrt{11}$

$$(3 - \sqrt{11})^2 - 6(3 - \sqrt{11}) - 2 = 0$$
$$9 - 6\sqrt{11} + 11 - 18 + 6\sqrt{11} - 2 = 0$$
$$0 = 0$$

$\therefore$ roots are $3 + \sqrt{11}$, $3 - \sqrt{11}$

NOTE. The roots of this equation are *irrational*. See section 124.

Oral Exercises

First find one half of the following, then square the result.

1. 12. **3.** $\frac{2}{3}$. **5.** $\frac{4}{5}$. **7.** $\frac{1}{2}$. **9.** $\frac{b}{2}$.

2. 18. **4.** $\frac{2}{5}$. **6.** $\frac{6}{7}$. **8.** $\frac{1}{4}$. **10.** $\frac{3b}{2}$.

Complete the square in

11. $x^2 + 2x = 8$ **13.** $x^2 + x = \frac{3}{4}$

12. $x^2 + 6x = 1$ **14.** $x^2 - 10x = 20$

Exercises

Solve each of the following equations by the method of completing the square, and check as directed by the teacher:

1. $x^2 + 4x = 21$.
2. $x^2 - 10x = 11$.
3. $x^2 - 2x = 35$.
4. $x^2 + 8x = 33$.
5. $x^2 + 12x = 28$.
6. $x^2 - 6x = 7$.
7. $x^2 + 12 = 8x$.
8. $x^2 + 10x = 11$.
9. $x^2 - 14x = 15$.
10. $x^2 - 2x = 15$.
11. $x^2 - 6x = 0$.
12. $x^2 - 12x = -11$.
13. $x^2 + 5x = \frac{11}{4}$.
14. $2x^2 - 9x + 10 = 0$.
15. $3x^2 - 10x + 3 = 0$.
16. $4x^2 - 21x + 20 = 0$.
17. $2x^2 - 7x = -3$.
18. $10x^2 + 3x = 1$.
19. $12x^2 - 7x - 10 = 0$.
20. $\dfrac{8}{x+2} = x$.
21. $x = \dfrac{4}{x+3}$.
22. $\dfrac{x}{3} + 4 = \dfrac{15}{x}$.
23. $\dfrac{x-3}{x} = \dfrac{2x}{x-2}$.
24. $\dfrac{x}{6} + 1 = \dfrac{12}{x}$.
25. $\dfrac{3}{x+1} - \dfrac{x}{4} = 3$.
26. $\dfrac{8}{x+6} = \dfrac{x-2}{16}$.
27. $x - \dfrac{1}{x} = 2 + \dfrac{1}{x}$.
28. $\dfrac{2x+1}{3x+5} = \dfrac{5x+1}{9x+7}$.
29. $\dfrac{4x+3}{x+1} - \dfrac{9x+5}{5x-1} = 0$.
30. $3x^2 - 6x - 5 = 0$.
31. $x^2 + 10 = 8x$.
32. $x(x+4) = 7$.
33. $9x^2 - 12x - 1 = 0$.
34. Is 4 a root of $\dfrac{x+2}{x+1} - \dfrac{3x}{2} = 6$?
35. Is 5 a root of $(x-1)(3x+1) = 12x + 4$?
36. Find out whether $2x^2 - 11$ does or does not equal 21 if $x = 7$.

143. Solving Literal Quadratics by Completing the Square.

Illustrative example:

Solve for x the equation $x^2 + 2ax - 3a^2 = 0$.

Solution:

$$x^2 + 2ax - 3a^2 = 0$$
$$x^2 + 2ax = 3a^2$$
$$x^2 + 2ax + a^2 = 4a^2$$
$$x + a = \pm 2a$$

$x + a = +2a$	$x + a = -2a$
$x = a$	$x = -3a$
$\therefore$ roots are $a, -3a$	
Check: $x = a$	*Check:* $x = -3a$
$a^2 + 2a^2 - 3a^2 = 0$	$9a^2 - 6a^2 - 3a^2 = 0$
$0 = 0$	$0 = 0$

Exercises

Solve for x by completing the square. Check.

1. $x^2 - 8mx + 7m^2 = 0$.
2. $x^2 - 6ax + 9a^2 = 0$.
3. $x^2 - 10ax + 21a^2 = 0$.
4. $x^2 - 4ax + 3a^2 = 0$.
5. $x^2 - 3bx - 10b^2 = 0$.
6. $x^2 + kx - 2k^2 = 0$.
7. $2x^2 + mx - 6m^2 = 0$.
8. $2x^2 + 9abx = 18a^2b^2$.
9. $x^2 + \frac{3x}{a} - \frac{10}{a^2} = 0$.
10. $x^2 - \frac{4x}{a} + \frac{1}{a^2} = 0$.
11. $\frac{24a^2}{x + 2a} = x$.
12. $c^2x^2 + cx - 6 = 0$.
13. $10x^2 - 3b^2 = -bx$.
14. $x^2 + bx + c = 0$.

144. Solving a Complete Quadratic Equation by the Formula. All complete quadratic equations may be changed to the form

$$ax^2 + bx + c = 0.$$

We may solve this equation by completing the square as follows.

Solution:

$$ax^2 + bx = -c.$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Since $ax^2 + bx + c = 0$ represents any quadratic equation, $\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ is the formula which represents the roots of any quadratic equation. This formula shows that the roots of any quadratic equation *depend* on the coefficients. The way in which this formula is used to solve quadratic equations is shown by the following

Illustrative example:

Solve $2x^2 + x = 6$.

Solution: Transpose every term to the left member as in the equation $ax^2 + bx + c = 0$.

We obtain, $2x^2 + x - 6 = 0$.

Then, a (the coefficient of x^2) $= 2$. $-4ac = 48$

b (the coefficient of x) $= 1$. $\sqrt{b^2 - 4ac} = \pm 7$

c (the term without x) $= -6$.

Substituting these values in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives us,

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4 \cdot 2(-6)}}{4}$$

$$= \frac{-1 \pm \sqrt{1 + 48}}{4}$$

$$= \frac{-1 \pm 7}{4}$$

$$= \frac{3}{2} \text{ or } -2.$$

Check with $\frac{3}{2}$: $2(\frac{3}{2})^2 + \frac{3}{2} = 6.$

$$2 \cdot \tfrac{9}{4} + \tfrac{3}{2} = 6.$$

$$\tfrac{9}{2} + \tfrac{3}{2} = 6.$$

$$6 = 6.$$

Check with -2: $2(-2)^2 + (-2) = 6.$

$$2 \cdot 4 - 2 = 6.$$

$$8 - 2 = 6.$$

$$6 = 6.$$

You should memorize the *quadratic formula* as given above. By its use *any* quadratic equation may be solved, and this method will often be found easier and more rapid than the method of completing the square. The above solution gives us the following

Rule. — 1. *Write the equation in standard form.*

2. *Write down the values of a, b, and c.*

3. *Substitute these values in the formula.*

4. *Perform all operations.*

5. *Check each root.*

Oral Exercises

Give each of the following in standard form.

1. $x^2 - 6x = -9$.
2. $x^2 - 5x = 6$.
3. $x^2 + 16 = -8x$.
4. $x^2 - 20 = 8x$.
5. $81 - 18x = -x^2$.
6. $-21 = 3x - x^2$.
7. State the values of a, b, and c in each of the above equations.

Exercises

By use of the quadratic formula solve each of the following equations. Check.

1. $x^2 - 8x + 15 = 0$.
2. $x^2 + 7x - 8 = 0$.
3. $x^2 + x - 42 = 0$.
4. $x^2 - 11x + 30 = 0$.
5. $2x^2 - x - 1 = 0$.
6. $6x^2 - x - 15 = 0$.
7. $4x^2 - 23x = 6$.
8. $15x^2 - 16x = 15$.
9. $8x^2 - 6x = -1$.
10. $3x^2 - 20x = 7$.

First remove parentheses and put in standard form.

11. $(x + 3)^2 = 10x + 5$.
12. $(5x - 1)(x + 6) = -25 + 5x$.
13. $(x + 7)(2 - x) = x^2 - 11$.
14. $(x - 3)^2 = (2x)^2 - 10x - 6$.
15. $(x - 2)(x - 3) = (2x - 3)^2 - 23$.

First clear of fractions and put in standard form.

16. $\frac{x}{2} + \frac{12}{x} = x - 1$.
17. $\frac{x - 2}{4} + \frac{12}{x} = \frac{x}{2}$.
18. $\frac{y^2}{6} + 1 = \frac{5y}{6}$.
19. $2 - \frac{2x + 1}{3} = \frac{1}{2x - 1}$.
20. $\frac{3}{x - 5} - 5 = \frac{-2x}{x - 3}$.
21. $\frac{5x - 7}{9} + \frac{14}{2x - 3} = x - 1$.

22. $\frac{7x}{6} = \frac{1}{2} - x^2$.

23. $\frac{1}{2}(x+1) - \frac{x}{3}(2x-1) = \frac{2}{3}$.

24. $\frac{2x}{5} + \frac{3x+9}{2} = \frac{20}{x} - 1$.

25. $2a^2x^2 - 7abx - 4b^2 = 0$.

Hint: In the formula $a = 2a^2$, $b = -7ab$, $c = -4b^2$.

26. $5x^2 - 16ax + 3a^2 = 0$.

27. $b^2x^2 - 8bcx = -7c^2$.

28. $m^2x^2 - mnx - 12n^2 = 0$.

29. $x^2 - (a+b)x + ab = 0$.

30. $x^2 + (5b - 2a)x - 10ab = 0$.

145. Finding the Roots of a Quadratic to the Nearest Tenth or the Nearest Hundredth.

Illustrative example:

Find the roots of $x^2 + 5x = -1$ to the nearest hundredth.

Solution: $x^2 + 5x = -1$ $\quad a=1, b=5, c=1, -4ac = -4$

$$x^2 + 5x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{25 - 4}}{2} = \frac{-5 \pm \sqrt{21}}{2}$$

$$= \frac{-5 \pm 4.582}{2} = -.209 \text{ or } -4.791$$

$$= -.21 \text{ or } -4.79$$

Check. An approximate check in the original is possible.

Exercises

Find the roots to the nearest tenth. Check as directed.

1. $x^2 + 6x + 1 = 0$.

2. $6x^2 + 3x = 5$.

3. $x^2 + 3x + 1 = 0$.

4. $x^2 - 8x - 6 = 0$.

5. $x^2 - 4x = 10$.

6. $x^2 - 6x = 8$.

7. $x^2 - 5x - 2 = 0$.

8. $x^2 - 3x - 7 = 0$.

9. $x^2 - 7x - 2 = 0$.

10. $\frac{x^2}{4} + x = \frac{1}{4}$.

11. $x^2 - .59x + .0828 = 0$.

12. $x^2 - .14x - .1395 = 0$.

Find the roots to the nearest hundredth:

13. $x^2 - x - 5 = 0$.

14. $x^2 - x - 7 = 0$.

15. $x^2 - 2x - 6 = 0$.

16. $.3x^2 + .75x = 1.4$.

17. $x^2 - .7x + .05 = 0$.

18. $x^2 - .07x = .3x - .012$.

19. $3x^2 - 6x = 2x - 1$.

Exercises

Solve these quadratics by any method.

1. $3x^2 + 7x - 2 = 0$.

2. $4x^2 = x + 8$.

3. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{82}{9}$.

4. $\frac{1+x}{2+x} - \frac{3-x}{5-x} = \frac{1}{6}$.

5. $\frac{t-2}{t+2} - \frac{t-1}{t+1} = 6$.

6. $\frac{z-5}{z+3} + \frac{z-8}{z-3} = \frac{3(1-z)}{z^2-9}$

7. $\frac{a}{x} + \frac{x}{a} = \frac{33a^2 - x^2}{ax}$.

8. $\frac{m}{x-n} + \frac{n}{x-m} - 2 = 0$.

9. $\frac{x - 4k + l}{l - k} = \frac{9l + x}{x}$.

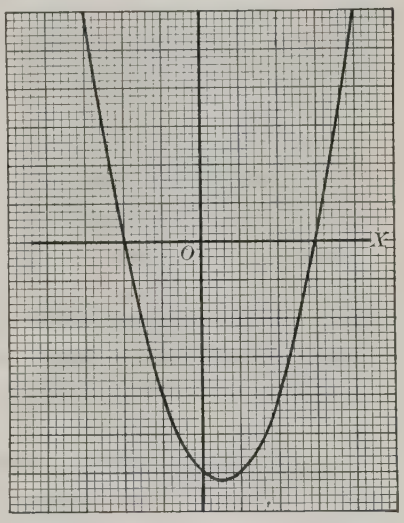
10. $\frac{3x-1}{x+2} + \frac{x-1}{3x+2} = \frac{5}{2}$

146. Graph of a Quadratic. — We may graph certain quadratic equations in much the same way that we made other functional graphs. Take, for example, the equation $y = x^2 - x - 6$. Here the value of y depends on the value of $x^2 - x - 6$, so we call y the function. Preparing a table

of corresponding values as we have done in previous graph work, we have

x	-3	-2	-1	0	1	2	3	4
y	6	0	-4	-6	-6	-4	0	6

We now locate the points as in graphing a linear equation and draw a smooth curve through them. This particular kind of curve is called a *parabola*. A quadratic function of x is always a parabola.



GRAPH OF THE QUADRATIC FUNCTION $x^2 - x - 6$

Exercises

From the graph

Find the value of x at each point where the graph crosses the *x-axis*.

Give the value of y at each of these points.

Solve the equation

$$x^2 - x - 6 = 0$$

by factoring.

The values of x which make the function $x^2 - x - 6$ equal to 0 are the roots of the equation $x^2 - x - 6 = 0$. From the graph you will observe that they are 3 and -2.

Exercises

Draw the graph of each of the following equations:

1. $y = x^2 - 2x + 1$.
2. $y = x^2 + 2x - 3$.
3. $y = x^2 + x - 6$.
4. $y = x^2 - 4x + 3$.

Solve graphically by putting y equal to 0 and check by substituting the roots in the equations:

5. Ex. 1 to 4.
6. $x^2 + 4x = 0$.
7. $x^2 - 4 = 0$.
8. $x^2 - 3x + 2 = 0$.
9. $2x^2 + 5x - 3 = 0$.
10. $x^2 = 0$.

11. Plot the graph of the function $x^2 - 2x - 7$.

12. What are the values of x in $y = x^2 - 2x - 7$ when y is 0? $+8$? -4? $+2$? -2?

13. From the graph of example 11 estimate the roots of $x^2 - 2x - 7 = 0$; of $x^2 - 2x - 7 = 8$; of $x^2 - 2x - 7 = -4$; of $x^2 - 2x - 7 = -2$.

14. Does $x^2 - 2x - 7$ increase or decrease as x increases from (a) -10 to 0? (b) From 0 to 3? (c) From 5 to 8?

15. Plot $y = x^2$ between $x = -5$ and $+5$. From the graph estimate $\sqrt{4}$; $\sqrt{10}$; $\sqrt{20}$.

147. Irrational Equations. — An irrational equation is one in which the unknown quantity occurs in the radicand of an irrational expression, as, for example,

$$4\sqrt{x-3} = x.$$

Such an equation may be simplified by squaring both members of the equation, to remove the radical signs, after which it may be solved by methods already shown.

In this connection you should notice that the processes of squaring a quantity and extracting its square root are exactly opposite processes, as are multiplication and division. For that reason when the square root of a quantity is extracted, and the result squared, the quantity obtained will be exactly the same as the one with which we started.

Hence, $(\sqrt{x-3})^2 = x - 3$. We therefore have the

Rule. — *To square a quantity under a radical of the second order simply omit the radical sign.*

Obviously in a radical of the third order the sign may be omitted in like manner when the expression is cubed.

Illustrative example:

Let us now solve the equation above, $4\sqrt{x-3} = x.$

Solution: Squaring both members we have, $16(x-3) = x^2,$

or $16x - 48 = x^2.$

Rearranging we then have, $x^2 - 16x + 48 = 0.$

Factoring, $(x-12)(x-4) = 0.$

Solving, $x - 12 = 0, \quad x - 4 = 0,$

$x = 12. \quad x = 4.$

Check with 12:

$4\sqrt{12-3} = 12.$

$4\sqrt{9} = 12.$

$4 \cdot 3 = 12.$

$12 = 12.$

Check with 4:

$4\sqrt{4-3} = 4.$

$4\sqrt{1} = 4.$

$4 \cdot 1 = 4.$

$4 = 4.$

Checking the solutions of irrational equations is of the greatest importance, as the numbers obtained from the solution do not always check. Such values are introduced in certain cases by the process of squaring. They are spoken of as *extraneous results,* and must be rejected. We must not forget that the final test of the root of an equation is the check. If the quantity satisfies the original equation, it is a root, otherwise it is not.

Oral Exercises

Solve:

1. $\sqrt{x} = 2.$
2. $\sqrt{x} = 3.$
3. $\sqrt{x+5} = 5.$
4. $\sqrt{x-2} = 4.$
5. $\sqrt{x-1} = 2.$
6. $\sqrt[3]{x-2} = 3.$
7. $\sqrt[4]{x-1} = 2.$
8. $\sqrt{x} - 3 = 0.$
9. $\sqrt{x} - 2 = 0.$
10. $\sqrt{x} - 5 = 0.$

11. $\sqrt{x} - 1 = 3.$
12. $\sqrt{x} - 2 = 2.$
13. $\sqrt{x} - 4 = 3.$
14. $\sqrt{x - 3} - 2 = 0.$
15. $\sqrt{x - 1} - 3 = 0.$
16. $\sqrt{x - 4} - 2 = 0.$
17. $\sqrt{x - 3} - 1 = 0.$
18. $\sqrt{x - 5} - 2 = 3.$
19. $\sqrt{x - 1} - 3 = 4.$
20. $\sqrt{x - 6} - 2 = 5.$
21. $\sqrt{5 - x} = 4.$
22. $3\sqrt{x} = 1.$
23. $\sqrt{2x} = \sqrt{5}.$
24. $\sqrt{4x} - 8 = 0.$
25. $2\sqrt{x} - 6 = 0.$

Exercises

Some of the equations which follow result in quadratic equations after being rationalized, while others reduce to equations of the first degree.

Solve and check each of the following equations:

1. $\sqrt{6x + 1} = 7.$
2. $3\sqrt{x + 1} = \sqrt{6}.$
3. $\sqrt[3]{2x - 1} = 5.$
4. $5\sqrt{10 - x} + 2 = 12.$
5. $2\sqrt[3]{3x - 9} = -4.$
6. $11 - 2\sqrt{5x - 4} = 3.$
7. $\sqrt{x^2 + 20} = 6.$
8. $\sqrt{x + 3} + 3 = x.$
9. $\sqrt{x + 4} + x = 8.$
10. $\sqrt{x + 25} = 5 - x.$
11. $\sqrt{x + 2} - \sqrt{16 - x} = 0.$
12. The time in seconds (t) for the complete vibration of a simple pendulum is given by the formula $t = \pi\sqrt{\frac{l}{32.2}}$, where l represents the length in feet of the pendulum and $\pi = 3.1416$. Solve this equation for the value of l, then find the length of a pendulum which vibrates in 2 seconds.
13. The velocity of a falling body, falling from rest, is obtained from the equation $v = \sqrt{2gs}$.

(a) Solve this for s (the space fallen in feet).

(b) Find s if $g = 32.2$ and $v = 128.8$ feet per second.

Number Problems Leading to Quadratic Equations

You should check each result in the problem as stated, and reject any which are meaningless, or which fail to satisfy the conditions of the problem.

1. The product of two consecutive numbers is 34 more than the next consecutive number. Find them.

2. The sum of two numbers is 13 and the sum of their squares is 85. What are the numbers?

3. There are two numbers whose difference is 5, and the difference of their squares is 45. What are they?

4. One number is two larger than a second number. The sum of the squares of these numbers is 19 more than the product of the numbers. Find the numbers.

5. The sum of the reciprocals of two consecutive even numbers is $\frac{13}{84}$. Find the numbers.

Hint: The reciprocal of a number is 1 divided by that number.

6. A boy set out 600 tomato plants. The number of plants in each row was 10 less than twice the number of rows. How many plants in each row?

7. A man bought two farms for \$4800 each. One farm contained 20 acres more than the other, and cost him \$8 less per acre. How many acres in each farm?

8. The number of things possible to happen may be denoted by $x(x - 1)$, or by 272. Find the value of x.

9. A boy was sent with \$2.10 to buy oranges; he found the price 3 cents per dozen higher than he expected, so he bought one half dozen fewer than he intended to buy and received 1 cent in change. What price did he pay for the oranges?

Boy Scouts at Camp

10. The expenses of a camping trip for a troop of Boy Scouts were 60 dollars. If there had been three more in the party, it would have cost each boy 1 dollar less. How many were in the party?

Geometrical Problems

11. The perimeter of a rectangle is 40 feet and its area is 96 square feet. Find the length and width.

12. The length of a rectangle exceeds its width by 7 feet, and the area of the rectangle is 120 square feet. What are its dimensions?

13. If three equal squares are placed side by side in a row, the result is a rectangle whose area in square rods is 15 times its perimeter in rods. Find a side of one of the squares.

14. The altitude of a certain triangle is twice its base. The area of the triangle is 64. Find the base and the altitude.

Hint: The area of a triangle is $\frac{1}{2}$ the product of the base and altitude.

15. The altitude of a certain triangle is 2 more than the base; the area is 60. Find the base.

16. The base of a triangle is 7 times the altitude. The area is 126. Find the base and the altitude.

17. If it takes 150 square inches of paper to cover a cube, what is the length of an edge of the cube?

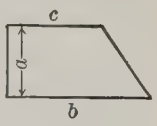
18. The area of a certain rectangle is 144 square feet. The perimeter is 60 feet. Find the length and the width.

Hint: Let x = the width.

Then, $30 - x$ = the length.

30 = one half of the perimeter.

19. The area of a trapezoid, as represented in the figure, is $\frac{a(b+c)}{2}$. If the upper base (c) is 2 more than the altitude (a), and the lower base (b) is 4 more than the altitude, find the bases and the altitude if the area is 54.



20. One base of a trapezoid is three times the altitude. The other base is 15. The area is 9. Find the altitude and the other base.

21. Suppose that we were to make a square uncovered box from a piece of metal. The metal is square, but square corners such as are indicated by the drawing are cut out. The remaining sides are bent up and then the edges are soldered. The box is to be 2 inches deep. It is to contain 128 cubic inches. How large shall the metal square be cut from which the box is to be made?

22. The dimensions of a rectangular box are represented by consecutive odd integers. The complete outside surface of the box is 46 square inches. What are the dimensions of the box?

23. If it requires 216 square inches of cardboard to make a cubical box, what is the length of an edge?

24. The altitude of a triangle is .5 of the length of the base. The area of the triangle is 6.25. Find the altitude and the base.

25. A man wishes to build a bin which will hold 9 tons of coal. The bin is to be 6 feet deep and 4 feet longer than it is wide. Allowing 40 cubic feet for 1 ton of coal, what must be the dimensions of the bin?

26. A piece of tin 9 inches by 12 inches is to be made into an open box with a base of 60 square inches by cutting equal squares from the four corners and then bending up the edges. What is the size, to the nearest tenth of an inch, of the square cut from each corner?

27. A bale of cotton contains 21 cubic feet. Its length is $4\frac{1}{2}$ feet, and its width is $\frac{1}{3}$ of a foot less than its thickness. Find its width and thickness.

28. A motion picture film 100 feet long contains a certain number of individual pictures. If each picture were $\frac{1}{4}$ inch shorter, the same film would contain 400 more pictures. How long is each picture?

29. A rectangular plot is 15 feet long and 10 feet wide. It is surrounded by a concrete walk of regular width. The area of the walk is equal to the area of the plot. How wide is the walk?

30. One leg of a right triangle is 1 foot longer than the other leg, and the hypotenuse is 2 feet longer than the shorter leg. Find the sides of the triangle.

Rate, Time, and Distance Problems

31. If a motorist makes 105 miles in 3 hours, how long will it take him at the same rate to go x miles? to go $x + 15$ miles?

32. A crew can row 8 miles down a stream and back again in 6 hours. If the rate of the stream is 1 mile an hour, find the rate at which the crew can row in still water.

A Close Race

33. A crew can row 4 miles down a stream and back again in 1 hour 50 minutes. If the rate of the stream is $2\frac{1}{2}$ miles an hour, find the rate of the crew in still water.

34. A man can row down stream 8 miles and back again in 5 hours. If the stream flows 2 miles an hour, what was his rate in still water?

35. Two steamers run between places 475 miles apart. One goes $\frac{1}{2}$ mile an hour faster than the other and takes $2\frac{1}{2}$ hours less time to make the trip. What are the rates of the steamers?

36. Two men start at the same time from the intersection of two roads, one driving north at the rate of 30 miles an hour, and the other east at the rate of 40 miles an hour. In how many hours will they be 250 miles apart?

148. Systems of Equations, One Linear and One Quadratic. — Till now all our quadratics have had only one un-

known. An equation in two unknowns is said to be *of the second degree*, or *quadratic*, if it contains the square of either unknown, *or the product of the two*. Examples are

$$x^2 + 2x + y = 10,\ x^2 + y^2 = 18,\ xy + 3y = 0.$$

In solving systems one of whose equations is quadratic and the other linear, we should eliminate by *substitution*, solving the linear equation for one unknown in terms of the other and substituting the result in the quadratic equation.

The method is shown by the following

Illustrative example:

Solve $2x + y = 7.$

$x^2 + xy = 12.$

Solution:

$y = 7 - 2x.$ Solving the linear equation for y.

$x^2 + x(7 - 2x) = 12.$ Substituting $7 - 2x$ for y in the quadratic equation.

$x^2 + 7x - 2x^2 = 12.$

$-x^2 + 7x - 12 = 0.$

$x^2 - 7x + 12 = 0.$

$(x - 4)(x - 3) = 0.$

$x - 4 = 0.$ $x - 3 = 0.$

$x = 4.$ $x = 3.$

$y = 7 - 2x.$

Hence, $y = 7 - 8$ or $7 - 6 = -1$ or 1.

The roots, therefore, are $\begin{Bmatrix} x = 4 \\ y = -1 \end{Bmatrix}$ or $\begin{Bmatrix} x = 3 \\ y = 1 \end{Bmatrix}$.

We should note that when $x = 4$, $y = -1$, and when $x = 3$, $y = 1$. The values of x and y must be carefully associated in pairs as indicated above, so that the pair forms a set of roots which will check in the systems. Such systems have, in general, two sets of roots. In order to check the roots of such equations, we should substitute the sets of roots in both original equations.

Exercises

Solve each of the following systems of equations, being careful to associate correctly the roots found. Check:

1. $x - y = 6.$
$x^2 + y^2 = 26.$

2. $x - y = 1.$
$x^2 + y^2 = 25.$

3. $x - y = 8.$
$x^2 + y^2 = 34.$

4. $x + y = 7.$
$x^2 - y^2 = 21.$

5. $x + y = 9.$
$x^2 - y^2 = 27.$

6. $x - y = 9.$
$x^2 + y^2 = 45.$

7. $x - y = 5.$
$x^2 + y^2 = 53.$

8. $x - y = 1.$
$xy = 12.$

9. $x - 3y = 1.$
$x^2 - y^2 = 15.$

10. $x - 4y = 6.$
$x^2 + 2xy = 36.$

11. $3x - y = 5.$
$x^2 + y^2 = 25.$

12. $2x - y = 2.$
$xy + 3y = 10.$

13. $4x = y + 3.$
$x^2 = 54 - 2y^2.$

14. $2x = y - 1.$
$x^2 - xy + y^2 = 19.$

15. $3x - 2y = 9.$
$4x^2 + 5y^2 = 36.$

16. $x^2 + xy = 12.$
$2x - 3y = 11.$

17. $xy - 3y^2 = -30.$
$4x + 3y = -5.$

18. $\frac{x^2}{2} - \frac{xy}{3} = 12.$
$x + y = 1.$

19. $\frac{x^2}{3} - \frac{y^2}{4} = -1.$
$2x + y = 10.$

20. *$x + y = 3.$
$3x^2 = y^2 + 14.$

21. $x^2 + y^2 + y = 10.$
$x + y + 1 = 0.$

22. $a - b\sqrt{6} = 0.$
$a^2 + b^2 = 14b.$

23. $3x^2 + y^2 = 40.$
$x - y\sqrt{3} = 0.$

24. The product of two numbers is $\frac{1}{3}$ and their sum is 4. Find the numbers.

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*Obtain roots to two decimal places.

Exercises

Problems leading to systems which may be solved by quadratic methods.

1. The difference of two numbers is 7 and the sum of their squares is 137. Find the numbers.

2. The sum of two numbers is 1 and the difference of their squares is 5. What are the numbers?

3. The product of two numbers is 72 and their difference is 14. Find them.

4. The quotient of two numbers is $\frac{3}{2}$ and their product is 216. What are the numbers?

5. The sum of two numbers is 8; the square of the first minus twice the square of the second equals 7. Find them.

6. If the sum of two numbers be added to the sum of their squares the result is 68, while if the difference of the same numbers be added to the difference of their squares the sum is 44. What are the numbers?

7. There is a rectangle whose perimeter is 46 feet and whose area is 112 square feet. What are its dimensions?

8. There is a rectangle whose perimeter expressed in feet is equal to its area in square feet. Its length is three feet greater than its width. Find its dimensions.

9. There is a rectangle whose perimeter is 18 inches. If its length is decreased 5 inches and its width is increased 12 inches, its area is doubled. Find its length and width.

10. The area of a rectangle is 20 square inches. If the length be increased by 4 inches and the width by 1 inch, the area will be 48 square inches. Find the sides.

11. The sum of the areas of two squares is 325 square feet. The side of one increased by the side of the other is 25 feet. Find the length of a side of each.

12. A tennis court for four players has an area of 2808 square feet. The perimeter of the court is 228 feet. Find the dimensions of the court.

13. The area of two squares is 117 square inches. Their perimeter is 60 inches. Find the length of a side of each square.

14. Construct a rectangle whose perimeter is 22 inches and whose area is 29.25 square inches.

15. The sum of the radii of two circles is equal to 17 inches. The sum of the areas of these two circles equals the area of a circle whose radius is 13 inches. Find the radii.

Hint: Area of a circle $= \pi$ times the square of the radius.

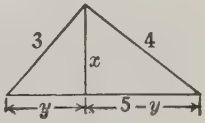
16. The square of the diagonal of a certain rectangle is 100. If two such rectangles are placed end to end, the square of the diagonal of the new rectangle is 292. Find the length and width of the original rectangle.

17. The hypotenuse of a right triangle is 20 inches. The sum of the other two sides is 28 inches. Find the lengths of those sides.

18. The perimeter of a rectangle is 17.6 inches. The area is 14.52 square inches. Find the length and width.

19. The two legs of a right triangle are equal. The sum of the squares of the two legs and the hypotenuse is 256. Find the sides of the triangle.

20. Find the altitude (x) of the adjoining triangle if the sides are 3, 4, and 5.



Hint: $x^2 + y^2 = 9.$

$x^2 + (5 - y)^2 = 16.$

21. Find the altitude to the side 12 of a triangle whose sides are 12, 14, and 15.

22. The front wheel of a wagon makes 100 more revolutions than the rear wheel in going 600 feet, but if the circumference of each is increased by 1 foot the front wheel will make only 50 more revolutions than the rear wheel. What is the circumference of each wheel?

Review

1. The perimeter of a circle is $2\pi r$ and the area is πr^2; express the area in terms of the perimeter.

2. A body which falls from rest is expected to follow the law denoted by the formula $s = 16.1\ t^2$, where s = distance in feet and t = time in seconds. If a stone dropped into a mine shaft were to strike the bottom in 5 seconds, how deep is the shaft?

3. Factor

(a) $x^2 + 10\,x + 25$.

(b) $6 - x - 15\,x^2$.

(c) $2\,k^2 - k - 1$.

(d) $x^{6m} + 13\,x^{3m} + 12$.

4. The following table shows the monthly sales of a salesman for a period of one year:

Month	Sales	Month	Sales	Month	Sales
Jan.	$675	May	$800	Sept.	$550
Feb.	700	June	700	Oct.	750
Mar.	600	July	600	Nov	800
Apr.	750	Aug.	400	Dec.	650

Draw a graph which will present the saleman's total sales from month to month. The last point should correspond with the total sales for the year.

5. Bradstreet reports the following analysis of the reasons why men fail, and the per cent of total failures due to each cause:

Incompetence . . .	38.2%	Failures of others . .	1.7
Lack of capital . .	30.3	Unwise credits . . .	1.3
Specific conditions .	11.3	Competition	1.1
Inexperience . . .	5.6	Extravagance . . .	1.1
Fraud	7.0	Speculation	.7
Neglect	1.7		100.0

Draw bar graphs to represent the above facts.

6. Using the following table:

p	0	1	2	3	4	5	6	7	8
c	0	6	12	18	24	30	36	42	48

Graph the equation $c = 6\,p$,

when c equals the cost and p equals the number of pounds of sugar at 6 cents a pound.

From the graph determine the cost of $9\frac{1}{2}$ pounds.

7. The diagonal of a rectangle is 15 feet. The perimeter is 42 feet. Find its dimensions.

8. Solve by the formula:

(a) $x^2 - \frac{11}{3}x + 2 = 0$. (b) $y^2 - \frac{1}{4}y = \frac{3}{20}$.

9. Reduce to lowest terms $\dfrac{x^2 - 5x - 14}{x^2 - 49}$.

10. A square board contains $\frac{5}{8}$ of a square foot area. Find a side to two decimals.

11. Will any real value of x make $(5 - 3x + x^2)^2$ negative?

12. What value of m will make $x^3 - 4x^2 + mx - 9$ exactly divisible by $x - 3$?

13. Define and give an example of

(a) A term.
(b) A monomial.
(c) A binomial.
(d) An exponent.

14. Solve the quadratic equation

$$(x + 1)(2x + 3) = 4x^2 - 22$$

by completing the square.

15. What value of y satisfies the equation

$$(y + 1)\left(\frac{y}{2} + 2\right) = (y + 2)\left(\frac{y}{2} + 1\right)?$$

16. Find the value of M, in

$M = \sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{1}{2}(a + b + c)$, when $a = b = c = 10$.

17. The perimeter of a rectangle, whose length is 5 inches longer than its width, is 34 inches. Find the dimensions.

18. In the formula for the area of a cube, $s = 6e^2$, state the changes in s when e is doubled. Trebled. Halved.

19. The area of a sphere is expressed by the formula $S = 4\pi r^2$. What change takes place in S when r is doubled? Trebled? Halved?

20. Draw a graph of the formula $S = 4\pi r^2$. Use $r = 7$, 14, 21, 28, and $\pi = \frac{22}{7}$.

21. If $a = \dfrac{b}{b - 1}$ and b is a positive number greater than 1, is a greater or less than 1?

22. If 42 times 38 is equal to $(40 + 2)(40 - 2)$, how would this principle apply to

(**a**) $56 \cdot 64$; (**b**) $78 \cdot 82$; (**c**) $23 \cdot 17$?

23. Solve $d = s\sqrt{2}$ for s, where s = the side of a square and d = the diagonal. What does s equal, if $d = \sqrt{32}$?

24. Find two consecutive even numbers whose product exceeds the square of the smaller by 24.

25. The formula for the use of lenses is $\dfrac{1}{D} = \dfrac{1}{F - d} - \dfrac{1}{f}$. Solve this for F; for D; for d.

26. From the equation $a - b = c - d$, we may obtain $a = c - d + b$. State the axiom that is used. What axiom is used to obtain $-b = c - d - a$ from the equation $a - b + d = c$?

27. Prove the following identity by multiplying out both members:

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 3(b + c)(c + a)(a + b).$$

28. Find the value of b in the equation $x = y^2 + 4y + b$ when $y = 1$ and $x = 5$.

29. Solve the equation

$$\sqrt{H} = L + \frac{7d}{8t} \text{ for } d. \quad \text{Also for } t.$$

30. Combine the following fractions into one simple fraction:

$$\frac{2x - 1}{x^2 - 8x - 9} - \frac{x - 3}{x^2 - 7x - 18}.$$

31. Solve for value of x and check:

$$\frac{4}{x - 3} - \frac{5}{x + 3} = \frac{3}{x^2 - 9}.$$

32. What is the median mark of nine graduates whose average standings for four years were 96, 78, 80, 65, 92, 79, 84, 87, and 90?

33. Find the value of 3 decimals of the following:

$$4\sqrt{7} + 7\sqrt{3} + 5\sqrt{2}.$$

34. Simplify:

$$\frac{x^2 - 9x + 18}{x - 3} \cdot \frac{x^2 - 9}{x^2 - 3x - 18}.$$

35. Draw a square. If its area is 64 sq. in., find a side. If A is 4225 sq. in.

36. Evaluate for the value of r in

$$r = \frac{1}{2}\sqrt{\frac{(a+b-c)(a-b+c)(-a+b+c)}{a+b+c}},$$

when $a = 3$, $b = 4$, and $c = 5$.

37. Is $(x-1)^3$ the same as $x^3 - 1$? Test, letting $x = 2$.

38. Define and illustrate:

(a) Radicand.

(b) An imaginary number.

(c) Similar radicals.

(d) A rational number.

(e) An irrational number.

(f) A surd.

(g) An irrational equation.

PART III
THROUGH NUMERICAL TRIGONOMETRY

CHAPTER XIII

RATIO, PROPORTION, VARIATION, DEPENDENCE

The principles of algebra which we have been studying form a foundation for more advanced work in mathematics. In this work one of the greatest time-savers and practical applications of what we have learned is *trigonometry*. It is based on the dependence of the sides and angles of triangles. Proper preparation for it requires a study of ratio, proportion, variation, and similar subjects.

149. Ratio. — The indicated quotient of two like quantities is called their *ratio*. For example, $\frac{3}{5}$ may be read "the ratio of 3 to 5," and $\frac{3 \text{ bu.}}{10 \text{ bu.}}$ may be read "the ratio of 3 bushels to 10 bushels."

A ratio is really a fraction. The *terms* of the ratio are the numerator and denominator of the fraction. Explain why the value of the ratio is not changed if both its terms are either multiplied or divided by the same number.

Hence the ratio $\frac{2\,x}{6\,x} = \frac{1}{3}$, and the ratio $\frac{10 \text{ qt.}}{2 \text{ qt.}} = \frac{5}{1}$ or 5.

A ratio is always a *number*, the quotient of the indicated division.

In order to express a ratio between two such quantities as 5 feet and 10 inches we must reduce them to the same unit. Thus 5 feet = 60 inches.

Hence $\frac{5 \text{ ft.}}{10 \text{ in.}} = \frac{60 \text{ in.}}{10 \text{ in.}} = \frac{6}{1}$ or 6.

Ratios may be expressed *decimally* or as *per cents.*

For example, the ratio $\frac{3}{5}$ may be written .6 or 60%. When we use per cents, the denominator of our ratio is 100.

Thus: $75\% = \frac{75}{100}$ or "the ratio of 75 to 100."

The ratio $\frac{3}{4}$ may be written 3 : 4. This is read "3 is to 4."

Exercises

Express each of the following ratios in fractional form, and reduce it to its lowest terms:

1. 6 to 10. **2.** 15 to 5. **3.** 8 to 6. **4.** 20 to 10. **5.** 24 to 12. **6.** 2 to 12. **7.** $1\frac{1}{2}$ to 3. **8.** $5\frac{1}{2}$ to 11. **9.** 15 to $7\frac{1}{2}$. **10.** 10 to $3\frac{1}{2}$. **11.** 4 qt. to 16 qt. **12.** \$20 to \$6.

13. 5 yd. to 40 yd.

14. 20 miles to 60 miles.

15. 30 lb. to 20 lb.

16. 4 oz. to 1 lb.

17. 5 ft. to 40 in.

18. 30 days to 5 weeks.

19. \$12 to 75¢.

20. 2 rd. to 11 ft.

21. 6 hours to 2 days.

22. 12 feet to 3 yards.

23. 15 inches to 1 yard.

24. 2 da. to 1 hr. 30 min.

25. 1 gal. 2 qt. to 6 gal.

26. .4

27. .8 **28.** .15 **29.** .25 **30.** .65. **31.** .125 **32.** 5% **33.** 20% **34.** $33\frac{1}{3}\%$. **35.** $66\frac{2}{3}\%$ **36.** 80% **37.** 48% **38.** 150%

39. Represent two numbers having the ratio 2 to 5.

Solution: If the first number is $2n$, the second number will be $5n$, for $\frac{2n}{5n} = \frac{2}{5}$.

40. Represent by use of n two numbers having the ratio 1 to 2; the ratio 4 to 7; the ratio 3 to 10.

41. By use of n represent 3 numbers having the ratio $1:3:5$; $2:5:8$; $4:7:12$.

42. What number has the same ratio to 8 as the ratio of 3 to 6?

43. The dimensions of a rectangle are denoted by $2x$ and $5x$. What is the ratio of its length to its width? Of its width to its length? Of its length to its perimeter? If its width is 12 inches what is its length? Its perimeter? What is the ratio of its width to its perimeter? Can we find a ratio of its length to its area? Why?

150. Proportion. — A *proportion* is a statement of the equality of two ratios.

Illustrative example:

$$\tfrac{3}{4} = \tfrac{75}{100} \text{ or } 3:4 = 75:100.$$

This may be read "3 is to 4 as 75 is to 100." We see from this example that a proportion is a very simple type of a fractional equation.

151. The Terms of a Proportion are the four quantities which compose it. They are sometimes called *proportionals*. In a proportion, the second and third terms are called *means* and the first and fourth *extremes*. Thus in the above proportion 4 and 75 are means while 3 and 100 are extremes. You will note that $4 \cdot 75 = 3 \cdot 100$. It is an important principle of proportion that *the product of the means equals the product of the extremes*. This may be proved by taking the proportion $\frac{a}{b} = \frac{c}{d}$ and clearing it of fractions by multiplying both ratios by bd, the lowest common multiple of the denominators. The result is $ad = bc$; a and d are extremes while b and c are means.

This principle enables us to find a missing term of a proportion when three terms are known.

Illustrative example:

Find x in the proportion $\frac{8}{20} = \frac{x}{30}$.

Solution: $20\,x = 8 \cdot 30$. By the above principle.

$20\,x = 240$.

$x = 12$.

Check: $\frac{8}{20} = \frac{12}{30}$, for each ratio reduces to $\frac{2}{5}$.

Exercises

Select the ratio at the right of the single line which makes a true proportion with the given ratio at the left of that single line:

1. $\frac{3}{4}$	$\frac{5}{12}, \frac{2}{3}, \frac{12}{20}, \frac{9}{12}$	4. $\frac{30}{3}$	$\frac{1}{10}, \frac{15}{2}, \frac{10}{1}, \frac{10}{3}$
2. $\frac{15}{18}$	$\frac{2}{3}, \frac{3}{5}, \frac{5}{6}, \frac{7}{12}$	5. $\frac{5\,x}{2\,x}$	$\frac{10\,n}{4\,n}, \frac{8\,y}{6\,y}, \frac{15}{4}, \frac{10}{2}$
3. $\frac{6}{2}$	$\frac{4}{3}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}$		

Solve each of the following proportions:

6. $\frac{x}{6} = \frac{5}{12}$.

7. $\frac{3\,x}{10} = \frac{12}{35}$.

8. $\frac{2\,x}{9} = \frac{5}{3}$.

9. $\frac{15}{8} = \frac{2\,x}{40}$.

10. $\frac{x-1}{6} = \frac{x+1}{10}$.

11. $\frac{11-x}{x} = \frac{9}{2}$.

12. $6 : 5\,x = 3 : 15$.

13. $4 : x = 12 : (x+12)$.

14. $(3\,x+1) : 2\,x = 4 : 2$.

15. $(5\,x-3) : (x-1) = 7 : 5$.

16. $9\,x : 4 = (2\,x-1) : 1$.

17. $(x+2) : (x+1) = (x+3) : (x+5)$.

18. Find two numbers in the ratio of 1 to 3 whose sum is 36.

Hint: Denote the numbers by x and $3\,x$.

19. There are two numbers in the ratio 5 to 2 and their difference is 30. What are they?

20. One number is 24 more than another and the ratio of the numbers is 2 : 5. Find them.

21. The sum of two numbers is 72 and their ratio is 5 to 1. Find the numbers.

22. A man willed $16,000 to his wife and son. The ratio of the wife's share to the son's share was 5 : 3. How much did each receive?

23. Two adjacent angles together make 180°. The ratio of the larger to the smaller is 3 to 2. How many degrees in each angle?

24. The sum of the three angles of a triangle is 180°. If these angles have the ratio 1 : 2 : 3, find the number of degrees in each.

25. Solve the above problem if the ratio of the angles is 1 : 1 : 2. If the ratio is 2 : 3 : 7.

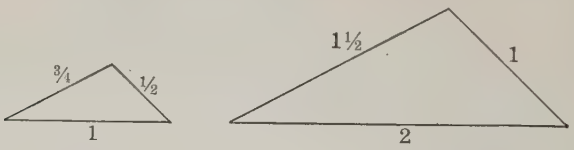
26. The width of a rectangle is to its length as 2 : 5. Its perimeter is 56 inches. What are the dimensions?

27. Two numbers are in the ratio 2 to 3. After 6 has been added to each they are in the ratio 3 to 4. Find them.

28. A ratio equal to $\frac{5}{8}$ may be changed to $\frac{4}{7}$ by subtracting 3 from its numerator and denominator. Find that ratio.

152. Similar Triangles. — Triangles whose sides are proportional are called *similar*. This means that they have exactly the same shape, one being only an enlarged or reduced copy of the other. The corresponding angles of similar triangles are equal.

The illustration shows a pair of similar triangles. The sides of one triangle are $\frac{1}{2}$ inch, $\frac{3}{4}$ inch, and 1 inch. The



corresponding sides of the other triangle are 1 inch, $1\frac{1}{2}$ inches, and 2 inches. Show that the corresponding sides are proportional.

The use of similar triangles enables us to find the lengths of lines without actually measuring them. It is one method of *indirect measurement.*

Illustrative example:

The sides of a triangle are 3, 4, and 6. In a similar triangle the side corresponding to 3 is 2. Find the other sides.

Solution: Call the sides to be found y and z.

Then $3:2 = 4:y$; hence $y = 2\frac{2}{3}$.

And $3:2 = 6:z$; hence $z = 4$.

Exercises

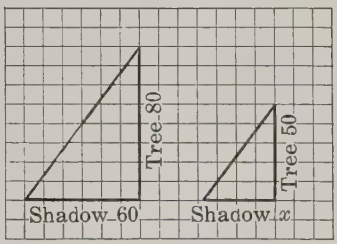
In examples 1 to 6, a, b, and c are the sides of a triangle and a' (a prime), b', and c' are the corresponding sides of a similar triangle.

1. $a = 6, b = 8, c = 10.$ $a' = 12.$ Find b' and c'.
2. $a = 4, b = 5, c = 7.$ $a' = 2.$ Find b' and c'.
3. $a = 10, b = 5, c = 6.$ $c' = 9.$ Find a' and b'.
4. $a = 4, b = 10, c = 12.$ $c' = 8.$ Find a' and b'.
5. $a = 12, b = 15, c = 20.$ $b' = 20.$ Find a' and c'.
6. $a = 20, b = 25, c = 40.$ $b' = 75.$ Find a' and c'.

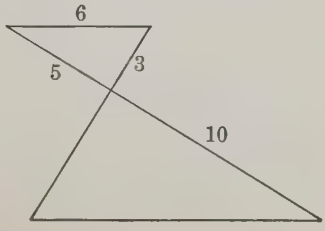
TREES AND SHADOWS

7. At a certain hour a tree 80 feet high casts a shadow 60 feet long. At the same time, how long a shadow was cast by a tree 50 feet high?

8. Find the height of a tree which casts a shadow 30 ft. long when a post 5 ft. high casts a shadow 2 ft. long.



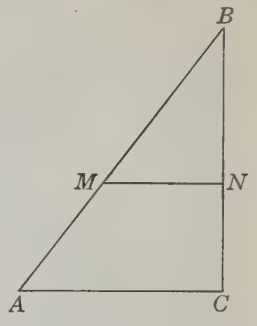
9. In the diagram the triangles are similar. Find the missing sides of the larger triangle.



10. The gable of my garage is a triangle whose slanting sides are each 12 feet and the base 20 feet. In a photo-

graph of this the base is 1 inch. How long are the sloping sides?

11. In the diagram the triangle MBN is similar to the triangle ABC. $AB = 10$ inches; $BC = 8$ inches, $AC = 6$ inches. $BM = 6$ inches. How long are BN and MN?



153. Constants and Variables. — A *constant* is a quantity whose value does not change in a given problem. A *variable* is a quantity thought of as changing in the problem. For example, if you leave your home on a train running uniformly 40 miles an hour, the rate of the train is constant, that is, it does not change. It is always 40. The distance you go will depend on the time you are on the train. That is variable. The following table shows how the distance varies with the time.

$r = 40$ (constant)

t (hours)	$\frac{1}{2}$	1	2 →	→ 3	5	8
d (miles)	20	40	80 →	→ 120	200	320

Now notice that any two values of t with the corresponding values of d form a proportion.

For example, $2 : 3 = 80 : 120$. As shown by the arrows.

Also $5 : 8 = 200 : 320$. Etc.

154. Direct Proportion : Direct Variation. — In the above example we may say: "The distance is directly proportional to the time." We may also say: "The distance varies directly as the time."

One variable quantity is directly proportional to another when the ratio of any two values of the one is the same as the ratio of the corresponding values of the other.

If you study the above table you will see that every value of d equals 40 times the corresponding value of t.

Hence $$d = 40\,t, \text{ or } \frac{d}{t} = 40.$$

One variable quantity varies directly as another when one equals a constant times the other, or when their quotient is constant. Hence, every direct variation can be expressed as an equation of the above type.

Exercises

1. If I ride for 3 hours in an airplane, the distance I go varies directly as the rate. (Assume a uniform rate for the whole trip.) Make a table like the above for the following rates: 100 miles an hour, 125, 150, 160, 200. Write 3 direct proportions resulting from the table. Also write the equation connecting d with r.

2. The perimeter (p) of a square varies as its side (s). Make a table of related values of p and s. Write a proportion and the resulting equation.

Similarly treat each of the following:

3. The cost of a dozen oranges varies as the price of one orange.

4. The amount a boy will earn in a week varies as his daily wage.

5. The amount of grain a farmer will feed his cattle in a week varies as the number of cattle, if he feeds each animal 1 peck a day.

6. If a man saves \$150 a year the amount of his savings varies as the number of years he has been saving at that rate.

155. Inverse Proportion: Inverse Variation.—If you were asked the question, "How long will it take you to walk 30 miles?" you would, no doubt, reply, "That depends on how fast I walk." The following table shows the relation between the rate and time, r and t.

$d = 30$ (constant)

r	1	2	3	5	6	10	15	30
t	30	15	10	6	5	3	2	1

This table means that if the rate is 1 mile an hour, the time required will be 30 hours; if the rate is 2 miles an hour, the time required will be 15 hours, and so on.

Here note the proportions as shown by the arrows, $1:2 = 15:30$. The ratio of the two values of r equals the ratio of the corresponding values of t *inverted.* Similarly $10:15 = 2:3$ etc. In this case we say "t is inversely proportional to r," or "t varies inversely as r."

Two variable quantities are inversely proportional if their product is constant. In the above example the product of r and t is always 30.

Exercises

1. A man rode 400 miles on a train traveling r miles an hour for t hours. Make a table showing values of r and t. Show the inverse proportion. State the constant product of r and t.

2. The area of a rectangle is 24 square inches. Then its length varies inversely as its width. Show this by a table.

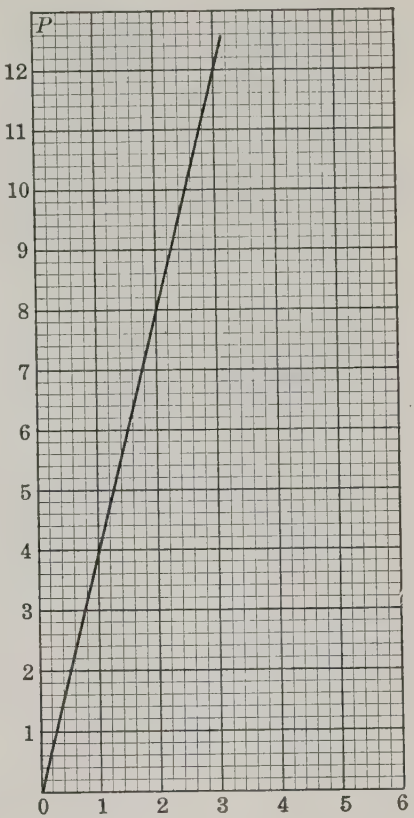
3. A man earned \$1000. The time required is inversely proportional to his daily wages. Construct the table.

4. A man saves \$600 in t years by saving d dollars a year. t varies inversely as d. Show by a table. What is the constant product of d and t?

156. Variation Shown by Graphs.—The relation between two changing quantities or variables may be shown by a graph. This has already been illustrated by the formula graph. For example, the graph on page 35 shows that s increases as t increases and that s increases more rapidly than t.

In the formula $p = 4\,s$, p varies directly as s because p is equal to s multiplied by a constant 4. The graph of this formula is a straight line. The graph shows that for any value of s the value of p is 4 times as great.

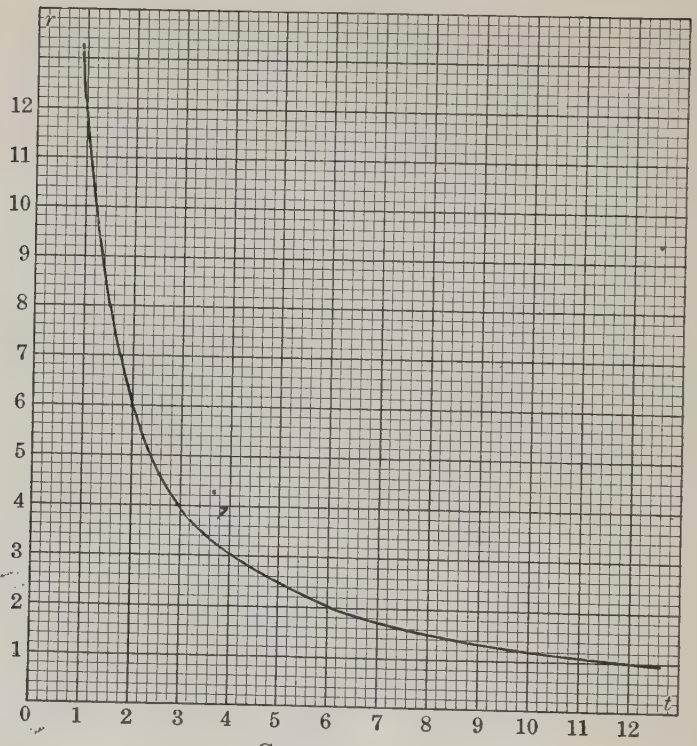
The graph of every direct variation is a straight line.



Graph of $p = 4\,s$

If I take a journey of 12 miles, and my average rate is denoted by r and the time necessary for the trip by t, then $rt = 12$.

In this case r varies inversely as t because the product of r



GRAPH OF $rt = 12$

and t is equal to a constant 12. This graph is a curved line called a *hyperbola*. Read from the graph the value of r if $t = 1, 2, 3, 4, 6, 8, 12$.

Exercises

1. In the formula $C = 2\pi r$ or $C = \frac{44}{7} r$, C varies directly as r. Graph the formula and from the graph read the values of C when $r = 1, 2, 3\frac{1}{2}, 7$.

2. In the formula $A = 10\,b$, A varies directly as b. Draw the graph and read the values of A when $b = 1, 2\frac{1}{2}, 3, 5$.

3. In the formula $d = \frac{n}{2}$, d varies directly as n. Draw the graph and read the values of d when $n = 1, 2, 3, 6, 10$.

4. In the formula $c = .25\,g$, c varies directly as g. Draw the graph and read the values of c when $g = 1, 4, 6, 8, 12$.

5. In the formula $i = .06\,p$, i varies directly as p. Draw the graph and read the values of i when $p =$ \$50, \$100, \$120, \$250.

6. In the formula $bh = 8$ or $h = \frac{8}{b}$, h varies inversely as b. Draw the graph and read the values of h when $b = 1, 2, 4, 6, 8, 12$.

7. In the formula $xy = 3$ or $y = \frac{3}{x}$, y varies inversely as x. Draw the graph and read the values of y when $x = 1, 1\frac{1}{2}, 2, 3, 6$.

8. In the formula $rp =$ \$60, r varies inversely as p. Draw the graph and read the values of r when $p =$ \$500, \$600, \$1000, \$2000.

157. Dependent and Independent Variables. — In much of our work we have seen how one variable quantity depends for its value on another. For example, if the rate is constant, the distance *depends* on the time; if time is constant, the distance *depends* on the rate; the perimeter of a square *depends* upon the length of the side; the amount of simple interest for one year *depends* on the rate per cent, and so on. Throughout the book we have frequently seen the idea of dependence expressed.

The quantity upon which the value of the other quantity depends is called the *independent variable*, while the quantity which depends upon it is called the *dependent variable*.

158. Functions. — Another name for a dependent variable is a *function,* as we learned in section 7. In other words, it is an unknown quantity which can be found as soon as we learn the value of the quantity of which it is a function. Thus, the distance a train will go in five hours is a function of its rate, because as soon as we learn the rate, we can find the distance. The area of a circle is a function of its radius, and the expression $2x^2 + x - 1$ is a function of x, because its value *depends* upon the value of x. For these reasons dependence is sometimes called *functionality.*

Exercises

Complete each of the following statements.

1. The value of the expression $\frac{x^2}{2}$ is a function of . . .
2. The value of $3y + 6$ is a function of . . .
3. The area of a square is a function of . . .
4. The circumference of a circle is a function of . . .
5. The distance I can ride my bicycle in 4 hours is a function of . . .
6. The simple interest \$250 will earn at 6% is a function of . . .
7. The interest one can receive in 8 years at 5% is a function of . . .
8. The area of a rectangle 3 feet long is a function of . . .
9. The amount I can earn in 6 days is a function of . . .
10. The height of the mercury in a thermometer is a function of . . .

Sometimes a variable depends for its value on two or more quantities. For example, if neither the length nor the width of a rectangle is a constant, its area depends on both; if neither rate nor time is constant, distance depends upon both.

Exercises

State two or more quantities upon which each of the following depends.

1. The perimeter of a rectangle.
2. The amount you can earn in a vacation.
3. The amount of interest your money will earn.
4. The number of cubic feet in a rectangular box.
5. The distance you will run in a race.
6. The time necessary to walk to school.

Walking to School

7. The expression $2x^2 + 3xy - y^2$.
8. State several things upon which you think your knowledge of algebra depends.

159. Functions in Formulas and Equations. — A formula, which shows the velocity (v) of a falling body (starting from rest) at the end of t seconds is $v = 32t$.

Here v is a function of t because its value *depends* on the value of t. Find v if $t = 1, 2, 3, 5, 8$.

In the equation $y = 3x - 1$, y is a function of x because y can be found if x is known.

Find y if $x = 1, 2, 3, 5, 6, 10$. If I solve the same equation for x, thus, $3x = y + 1$ or $x = \frac{y+1}{3}$, then x is thought of as a function of y. Find x if $y = 2, 5, 8, 20$. If the equation is written $3x - y = 1$, the same functional relationship of x and y is *implied.*

Illustrative example I:

What function of x is implied in the equation $2y+3x=10$?

Solution: Adding $-3x$ to both members of the equation we have $2y = 10 - 3x$.

Dividing by 2, $y = 5 - \frac{3}{2}x$.

Illustrative example II:

Express y as a function of x in $3x - 6y = 8$.

Solution: Adding $-3x$ to both members and dividing by -1

$$6y = 3x - 8$$

Dividing by 6, $$y = \frac{3x - 8}{6}$$

Or $$y = \frac{1}{2}x - \frac{4}{3}.$$

Exercises

Express x as a function of y in examples 1 to 5.

Express y as a function of x in examples 6 to 11.

1. $2x + y = 6$.
2. $x - y = 10$.
3. $2y + 6x = 9$.
4. $4x + 3y = 12$.
5. $2x - 7y = 10$.
6. $x^2 + y = 0$.

7. $x^2 - 4y = 10$.
8. $2x + y = 6$.
9. $4x - 5y = 20$.
10. $x^2 + x - y = 6$.
11. $2x^2 + 4x + 2y = 5$.

160. Fractional Functions. — We may solve the equation $xy = 10$ for y in terms of x and obtain $y = \frac{10}{x}$.

Here y is a *fractional function* of x.

Exercises

Express y as a function of x in each case.

1. $2xy = 5$.
2. $3xy + x = 6$.
3. $2xy - y = 10$. *Hint:* $y(2x - 1) = 10$.
4. $6xy + y = 4$.
5. $2xy + 3x - y = -2$.

161. Irrational Functions. — The equation $y^2 + 2x = 6$ implies that y depends upon x, that is, that y is a function of x. In order to express this function definitely we must solve the equation for y expressed in terms of x, thus

$$y^2 + 2x = 6$$
$$y^2 = 6 - 2x$$
$$y = \pm\sqrt{6 - 2x}.$$

In this case y is an *irrational function* of x.

Exercises

Express y as a function of x in each case.

1. $y^2 + 2x = 0$.
2. $y^2 - 2x^2 + 1 = 0$.
3. $y^2 + x^2 + 5x = 0$.
4. $y^2 - 6x + 4 = 0$.
5. $y^2 - 2x = 2$.
6. $y^2 + 3x - 10 = 0$.
7. $y^2 - 6 = 4x + 2$.
8. $\frac{y^2}{2} + \frac{2}{3} = x$.

Exercises

1. In the formula $C = \frac{5}{9}(F - 32)$, upon what does C depend? Find C if $F = 0, 32, 212, 10, 15, 20, 40, -20$.

2. In the formula $s = 16\,t^2$, upon what does s depend? Find s if $t = 1, 2, 3, 4, 6$.

3. In the formula $v = at$ if $a = 10$ always, upon what does v depend? Find v if $t = 3, 5, 6, 8$.

4. In the formula $A = bh$ if $b = 8$ always, upon what does A depend? Find A if $h = 1, 2, 3, 5, 10$.

5. In the formula $C = \frac{44}{7}r$ upon what does C depend? Find C if $r = 7, 14, 28, 49$.

6. In the formula $A = \frac{22}{7}r^2$ upon what does A depend? Find A if $r = 7, 14, 16, 20$.

7. In the equation $y = \dfrac{3x + 6}{2}$ find y if $x = 0, 1, 2, 3, 5, 10$.

8. In the equation $y = \frac{1}{2}x^2 + 2x$ find y if $x = 0, 1, 2, 4, 6$.

9. In the equation $2x + 3y = 10$, express y as a function of x. Express x as a function of y.

10. Express the equation $3x - 2y = 6$ so that x is a function of y, and find x if $y = 0, 2, 4, 6, 10$.

162. Graphs Showing Dependence. — We have seen in section 103 that the condition that makes possible a functional graph is that one of the quantities shall be a dependent of the other. For example, the graph of the equation $2x + 3y = 6$ shown on page 295 is a graph of the function $y = \dfrac{6 - 2x}{3}$.

The abscissa of any point on this line is a value of x and the ordinate of the point is the corresponding value of y or of the function of x. By studying that diagram state the value of the

function when

$y=\frac{6-2x}{3}$										
$x=$	0	1	2	4	-1	-2	-4	-5	$\frac{1}{2}$	$-\frac{3}{2}$

This function of x is called a *linear* function because its graph is a straight line.

Exercises

Show how one quantity depends upon the other in the following graphs:

1. Draw the graph of the function $y = \frac{x-2}{3}$.

2. Draw the graph of $y = \frac{-x+6}{4}$.

3. Draw the graph of $y = \frac{5-2x}{2}$.

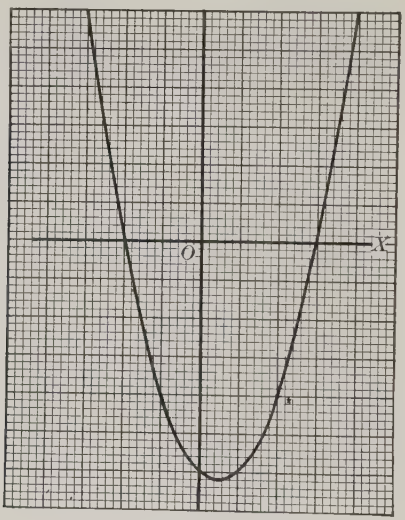
4. Draw the graph of $y = \frac{x-9}{3}$.

5. Draw the graph of $y = \frac{3}{2} + 3x$.

6. State by inspection of the graphs the corresponding values of y when $x = 0, \frac{1}{2}, \frac{3}{2}, 2, 3, 5, -1, -2, -\frac{5}{2}, -4$.

163. The Quadratic Function. — In section 146 we made a graph of a quadratic. Taking the quadratic function of x, as $x^2 - x - 6$, we called that function y. This gave us the equation $y = x^2 - x - 6$. It is plain that the value of y *depends* on that of x, or that y is the unknown quantity whose value we learn when we find that of x. From this equation

we prepared our table of values, plotted our points, and made our graph as shown below.



GRAPH OF THE QUADRATIC FUNCTION
$x^2 - x - 6$

164. How a Change in One Variable Changes the Other. — In the equation $y = bx + 1$ it is understood that b is a fixed constant, and that b, x, and y are all positive numbers. Then if x is increased bx is increased and so $bx + 1$ is increased also. Since $y = bx + 1$, we see that if x increases y increases. Also, if x decreases y decreases. This principle is true generally, if y is an integral function of x with positive coefficients.

However, in the equation $y = \frac{6}{x}$ if x is increased, y is decreased, for we make the fraction smaller if the denominator is made larger while the numerator is unchanged. Moreover, if x is doubled y is only half as large, while if x is halved y is

doubled. The same would be true if $y = \frac{a}{x}$ where a is any positive constant. That is, if x is multiplied by a positive number then y is divided by that number. Hence, if y is a fractional function of x with positive coefficients, we may find one variable increasing as the other decreases.

Exercises

(The letters are understood to represent positive numbers.)

1. $y = 3x$. What change takes place in y if x is increased, decreased, doubled, trebled, halved?

2. $y = \frac{bx}{2}$ where b is a positive constant. What change takes place in y if x is increased, decreased, doubled, trebled, halved?

3. $y = x^2$ and x greater than 1. What change takes place in y if x is increased, decreased? Answer the same question if x is less than 1.

4. $y = \frac{10}{x}$. What change takes place in y if x is increased, decreased, doubled, trebled, halved?

5. $y = \frac{a}{2x+1}$. If a is a positive constant, what change takes place in y if x is increased, decreased?

6. If $y = 2x + \frac{1}{2}$, what change takes place in y if x is increased, decreased?

7. If $y = 6 + \frac{2}{x}$, what change takes place in y if x is increased, decreased?

8. If $y = 2 + \cfrac{1}{3 + \frac{1}{x}}$, what change takes place in y if x is increased, decreased?

True-False Test

State which of the following are true and which are false:

1. The graph of the equation $2x-5y=12$ is a straight line.
2. The graph of the equation $2x+3y=0$ passes through the origin 0.
3. The graph of the equation $x=2$ is parallel to the X-axis.
4. The graphs of $x-y=2$ and $x-y=4$ are parallel lines.
5. The graphs of $5x+2y=-10$ and of $x+2=-\frac{2}{5}y$ are the same line.
6. The graph of a direct variation is a straight line.
7. The graph of an inverse variation is a straight line.
8. The graph of a quadratic function is always a parabola.
9. A parabola is the same as a hyperbola.
10. x^2+x+1 is a linear function of x.
11. π is a constant in the formula $C=\pi d$.
12. 7 is a constant in $7a^3b^2$.
13. There are only two variables in $d=rt$.
14. $\frac{1}{3}$ is a variable in $\frac{1}{3}y^2x$.
15. 9 is a constant in $9x$.
16. There are as many constants as variables in $F=1.8C+32$.
17. There are more constants than variables in $S=\frac{1}{2}at^2$. (a = a constant).
18. $m=dV$ has only two variables.
19. $A=\pi r^2$ has only one variable.
20. In $g=\dfrac{28f}{r}$ if f (the number of teeth in the front sprocket of a bicycle) increases and r (the number of teeth in the rear sprocket of the bicycle) is a constant, then g (the gear of the bicycle) increases.

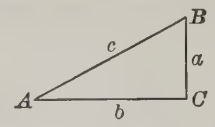
CHAPTER XIV

NUMERICAL TRIGONOMETRY

Our work on ratio, similar triangles, functions, and variation has prepared us for a study of *indirect measurement.* This is one of the most important and interesting branches of mathematics. By it one boy found that he could perform seemingly impossible measurements so easily that he called it "trick-onometry."

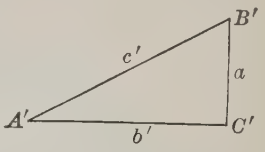
165. Trigonometry is a word derived from the Greek, meaning "triangle measurement." We shall see how the simple idea of ratio when applied to the sides of a triangle leads to this important method of *indirect measurement.* By its use we can solve a great variety of practical problems.

We shall deal mainly with the right triangle, as shown in this figure. The angles are usually denoted by the capital letters A, B, and C, and the sides opposite those angles by the corresponding small letters a, b, and c. We always put the letter C at the vertex of the right angle and mark the hypotenuse c. The hypotenuse is the side opposite the right angle; the other two sides are called *legs.*



166. Tangent of an Angle. — The ratio of the leg a to the leg b, that is, $\frac{a}{b}$ in the above figure, is called the *tangent* of the angle A, and is written "tan A." If now we have another right triangle $A'B'C'$, the tangent of the angle A' is in like manner $\frac{a'}{b'}$. When you come to study geometry,

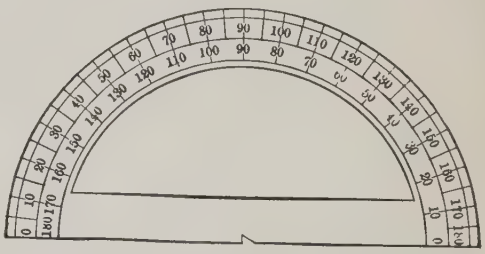
you will prove that if the angle A is equal to the angle A' then the proportion $\frac{a}{b} = \frac{a'}{b'}$ is always true, for the triangles are similar. In other words the ratio $\frac{a}{b}$ always has the same value when the angle A is unchanged in size, whether the triangle be large or small.



The tangent of one of the acute angles of a right triangle is the ratio of the leg opposite the angle to the leg adjacent to the angle.

What ratio represents the tangent of the angle B' in the above figure?

167. How Angles Are Measured. — The size of any angles which are drawn on paper may be found by the use of an instrument called the **protractor.**



This is done by putting the notch of the protractor at A, in the figure in section 165, and then by having its base-line coincide with AC. Then by sighting along the line AB, you can find the number of degrees in angle A.

Similarly, in section 166, you could find the size of angle A'.

Angles in the field are measured by an instrument called *a surveyor's transit.*

168. Measuring the Tangent of an Angle. — By using your protractor measure the number of degrees in the angle A in each of the three triangles on the squared paper. If you have learned to use this instrument accurately, you will find

45° in Fig. 1, 30° in Fig. 2, and 60° in Fig. 3. You will notice that in each of these figures the length of b is the same, 3 large squares. Now measure the length of the side a in

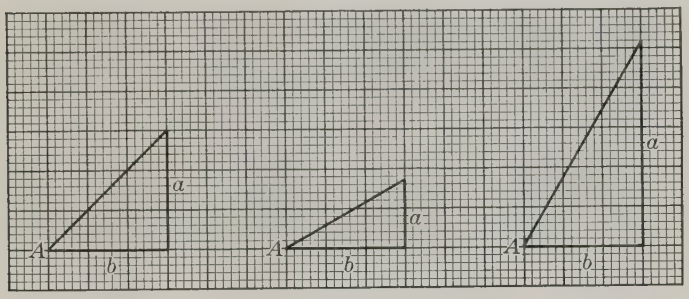


Fig. 1 Fig. 2 Fig. 3

each case. The pupils of one class in high school obtained the following results: Fig. 1, $a = 3$; Fig. 2, $a = 1.7$; Fig. 3, $a = 5.2$. Do you agree with them?

Then $\tan 45° = \frac{3}{3} = 1$, $\tan 30° = \frac{1.7}{3} = .57$, $\tan 60° = \frac{5.2}{3} = 1.73$. By this method, and in other ways also, we can compute the value of the tangent of any angle. The table on pages 450–454 shows that the correct values of these tangents to four decimal places are: $\tan 45° = 1.0000$, $\tan 30° = .5774$, $\tan 60° = 1.7321$.

169. Finding Tangents by Table. — Finding tangents of an angle from a table is spoken of as a *direct use of the table.* To find the tangent of an angle which is given exactly in the table we proceed by the following

Rule. — *Locate the number of degrees* (°) *and minutes* (′) *in the column so marked, then follow the horizontal line to the right until you come in the column headed tan.* There you will find the value given to four decimal places.

Illustrative example:

$$\tan 42° 40' = .9217, \tan 73° 10' = 3.3052.$$

If the given angle is not found exactly in the tables we must proceed as shown in the following

Illustrative example: Find $\tan 19° 24'$.

Solution: On page 451 we find $\tan 19° 20' = .3508$ and $\tan 19° 30' = .3541$. This is a difference of $10'$ in the angle. Now $19° 24'$ is .4 of the way from $19° 20'$ to $19° 30'$. The difference between .3508 and .3541 is .0033. Hence we multiply .0033 by .4, carrying the result to the nearest figure in the fourth decimal place; thus, $.4 \times .0033 = .00132 = .0013$. This is added to the tan of $19° 20'$ as a *correction*. Thus, $\tan 19° 24' = .3508 + .0013 = .3521$. This process is called *interpolation*.

Exercises

Find from the table the tangent of each of the following angles:

1. Tan 10°
2. Tan 16° 20′
3. Tan 40° 40′.
4. Tan 64° 30′.
5. Tan 32° 15′.
6. Tan 18° 45′.
7. Tan 52° 25′.
8. Tan 33° 35′.
9. Tan 5° 25′.
10. Tan 26° 12′.
11. Tan 29° 38′.
12. Tan 65° 44′.
13. Tan 12° 52′.
14. Tan 8° 56′.
15. Tan 24° 36′.

170. Inverse Use of the Table. — To find an angle of which the tangent is given, look in the column headed tan until the value is found. Follow this horizontal line to the left and the desired angle will be found in the column headed "Angle."

Illustrative example I: $\tan A = .4663$. Find the angle.

By the above rule $A = 25°$.

If the given value cannot be found exactly in the table, we proceed by the following

Rule. — *Locate the angle between two successive angles and write down the smaller one. Find the difference between the tangent of this angle and the one you are looking up. Also find the difference between the two tangents in the table, between which the one you are finding lies. Divide the first difference by the second, carrying the result to the nearest tenth. Add this number of minutes to the angle already written.*

Illustrative example II: $\tan A = .3089$. Find A.

Solution: We find the value .3089 in the first or tangent column on page 450. The corresponding angle is $17° 10'$. Hence angle $A = 17° 10'$.

Illustrative example III: $\tan A = 1.0492$. Find A.

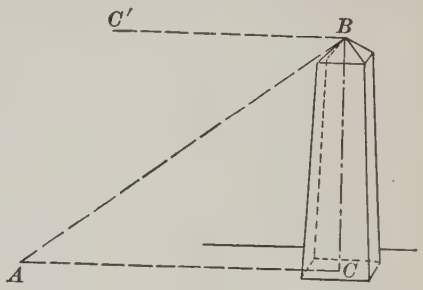
Solution: In the tangent column of page 452, we find $\tan 46° 20' = 1.0477$ and $\tan 46° 30' = 1.0538$ and the given tangent lies between these two. Hence the required angle lies between $46° 20'$ and $46° 30'$. The difference between 1.0477 and 1.0492 is .0015. The difference between 1.0477 and 1.0538 (called the *tabular difference*) is .0061. We now divide .0015 by .0061, obtaining .2. Now we add a correction of 2 minutes to $46° 20'$. Hence $A = 46° 22'$.

Exercises

Find the number of degrees and minutes in the angle A when the tangent of the angle in each of the following is:

1. .0963.	**6.** .7280.	**11.** 1.0956.
2. .1853.	**7.** .5600.	**12.** 1.3205.
3. 1.0235.	**8.** .4130.	**13.** 3.0350.
4. 3.3759.	**9.** .1625.	**14.** 4.8000.
5. 5.8708.	**10.** .9030.	**15.** .0422.

171. Angles of Elevation and Depression. — In determining the height or depth of objects above or below the horizontal plane, it is necessary to measure angles in a vertical plane. For example, if it were required to find the height of the monument in the diagram, it would be necessary to determine the angle BAC. If, however, we were stationed at B, the angle ABC' would do just as well, as these angles are equal.



An *angle of elevation or depression* is the angle between the line of sight of the observer and a horizontal line through the point of observation. When this angle is above the horizontal plane, it is called an angle of elevation, and when it is below it is called an angle of depression. In the diagram, angle BAC is an angle of elevation, and angle ABC' is an angle of depression.

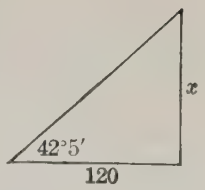
Both vertical and horizontal angles are measured in practice by a *surveyor's transit.*

172. Solving Problems by Tangents. — The examples which follow show how tangents are used in indirect measurement. You should draw a diagram to fit the problem in each case, as in the illustrative examples which follow.

Illustrative example I:

A boy standing on level ground notices that the angle to the top of a flagpole is 42° 45′. He measures the distance to the foot of the pole and finds it to be 120 feet. How high is the pole?

Solution: $\tan 42° 45' = .9217 + .5(.0054)$
$= .9217 + .0027$
$= .9244.$



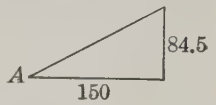
Then $\frac{x}{120} = \tan 42° 45'.$

$\frac{x}{120} = .9244.$

$x = 120(.9244) = 110.9$ feet. *Answer.*

Illustrative example II:

A tree is 84.5 feet high. What is the angle of elevation of its top when viewed from a point on level ground 150 feet from its foot?



Solution: $\tan A = \frac{84.5}{150} = .5633.$

Hence $A = 29° 20' +$ correction.

$\frac{.0014}{.0039} = .4.$

Therefore $A = 29° 24'.$

Exercises

1. If $b = 75$ feet and angle $A = 84° 15'$, find the length of a.

2. If $a = 120$ feet and angle $A = 48° 10'$, find the length of b.

3. If $a = 91$ feet and $b = 250$ feet, find angle A.

4. Standing on the sidewalk 200 feet from our city hall, I saw that the angle of elevation of the top of its tower was $38° 16'$. How high was the tower?

5. An airplane is directly over a town which is known to be 6 miles from the observer. The angle of elevation of the plane is found to be $21° 10'$. How high was the plane?

6. If a right triangle is constructed whose sides inclosing the right angle are 5 inches and 12 inches, then the greater acute angle is often taken as $\frac{3}{4}$ of a right angle. How nearly correct is this? The greater angle is opposite the greater side.

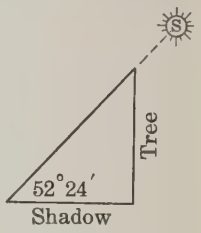
7. When an airplane is 2500 feet high, the angle of depression of a building is 77° 25′. How far is the building from the point directly beneath the airplane?

8. A dam 50 feet high makes a lake whose length is 4520 feet. At what angle does the ground slope?

9. A lighthouse is 150 feet high. Its angle of elevation as viewed from a ship at sea is 16° 12′. How far from the lighthouse is the ship?

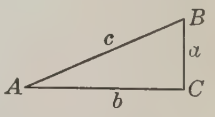
10. When the angle of elevation of the sun is 52° 24′, how long will be the shadow cast by a tree 60 feet high?

11. If a tree 75 feet high casts a shadow 40 feet long, what is the angle of elevation of the sun?



12. A man in a boat in the middle of a river a mile wide observes the angle of elevation of a cliff on the bank to be 12° 12′. How high is the cliff?

173. The Sine of an Angle. — The tangent is not the only ratio connected with the right triangle which is useful in measurement. Another is the *sine* of the angle. The sine is the ratio of the side opposite the angle to the hypotenuse. For example, in the diagram, the sine of A (written $\sin A$) $= \frac{a}{c}$. What ratio equals $\sin B$?



In the table on pages 450 to 454 sines will be found in the second column. Sines are read from the table in the same

manner as tangents and interpolation is performed in exactly the same way. Notice that the sine of an angle is never greater than 1.

Exercises

Find from the table:

1. sin 16° 40′.	**5.** sin 65° 25′.	**9.** sin 56° 56′.
2. sin 39° 10′.	**6.** sin 19° 44′.	**10.** sin 4° 48′.
3. sin 77° 30′.	**7.** sin 23° 12′.	
4. sin 84° 50′.	**8.** sin 39° 24′.	

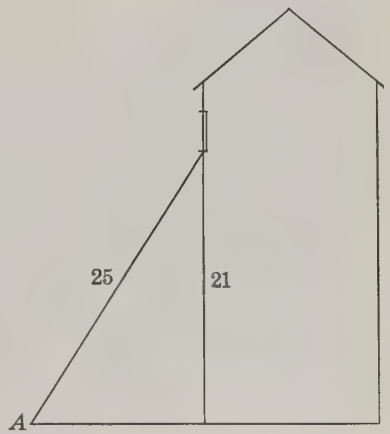
Find the angle whose sine in each case is:

11. .7898. **13.** .5075. **15.** 6873. **17.** .4932. **19.** .7770.
12. .9304. **14.** .3529. **16.** .8725. **18.** .3183. **20.** .9006.

174. Problems Solved by Sines. — The following example shows how the sine of an angle is used in problems.

Illustrative example:

A ladder 25 feet long is set up so that it just reaches a window 21 feet from the ground. What angle does it make with the ground?



Solution: $\sin A = \frac{21}{25} = .8400.$

From the table $\sin 57° = .8387.$

$\sin 57° 10′ = .8403.$

Hence $A = 57° + \text{correction}.$

$13 \div 16 = .8.$

Hence correction to be added is 8′.

Therefore $A = 57° 8′.$ *Ans.*

Exercises

1. One of the angles of a right triangle is 74° 15′ and the hypotenuse is 24. How long is the side opposite that angle?

2. A ladder 30 feet long is set up against a house so that it makes an angle of 66° 30′ with the ground. How far up the side of the house will it reach?

3. A kite is 150 feet above the ground, and the kite string is 200 feet long. If the string is straight, what angle does it make with the ground?

A Concrete Road Up a Hill

4. If a concrete road up a hill rises 17.5 feet for 100 feet of length of road, what is the angle of elevation of the road?

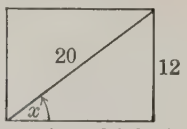
5. A boy standing on level ground and looking at a bird's nest in a tree 54 feet from the ground, notices that the angle of elevation of the nest is 62° 12′. How far is he from the nest?

6. A wedge is in the shape of a right triangle. How wide

is the wedge if its length measured on the hypotenuse is 20 inches and the angle of its edge is 10° 20′?

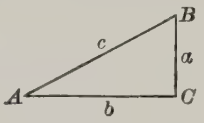
7. If a railroad up a mountain rises 1198 feet for 1 mile of length of track, what is the angle of elevation of the track?

8. In the rectangle in the diagram one side is 12 and the diagonal is 20. How large is the angle x?



175. The Cosine of an Angle. — A third ratio which is useful in solving certain problems is the *cosine*. The cosine is the ratio of the side adjacent to the angle to that of the hypotenuse. Cosine is abbreviated cos. Thus in the diagram

$$\cos A = \frac{b}{c}. \quad \text{Cos } B = \ ?$$



Cosines of angles are found in the third column of the table on pages 450 to 454. Cosines are read from the table in the same manner as tangents and sines, except that in interpolation the correction is subtracted. This is because the cosine becomes smaller as the angle increases.

Illustrative example: Find cos 37° 12′.

Solution: cos 37° 10′ = .7969.

cos 37° 20′ = .7951.

The tabular difference = .0018.

37° 12′ is .2 of the way from 37° 10′ to 37° 20′.

$.2 \times .0018 = .0004$.

Subtract this from .7969 and we obtain cos 37° 12′ = .7965.

Exercises

Find from the table:

1. cos 62° 40′. **3.** cos 13° 20′. **5.** cos 12° 32′.

2. cos 81° 10′. **4.** cos 29° 15′. **6.** cos 77° 48′.

Find A from the table in each case:

7. $\cos A = .8241$.

8. $\cos A = .3145$.

9. $\cos A = .4020$.

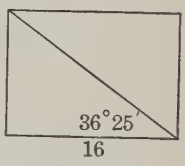
10. $\cos A = .5190$.

11. $\cos A = .6200$.

12. $\cos A = .7870$.

13. A ladder 30 feet long is set up against a building so that it makes an angle of 72° 18′ with the ground. How far is its foot from the building?

14. One side of a rectangle is 16 feet. The diagonal makes an angle of 36° 25′ with that side. How long is the diagonal?

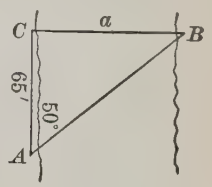


15. A boat sails in a direction 24° 32′ west of north. How far must it go in order to reach a point 50 miles farther north than the starting point?

16. An airplane flies in a direction 79° 20′ east of north. How far must it go to reach a point 185.1 miles farther north than the point from which it started?

The following examples are to be solved by the use of a suitable function.

17. The distance AC on level ground along a stream is 65 feet. The angle A is measured and found to be 50°. The triangle ACB is a right triangle. Find the distance across the stream.



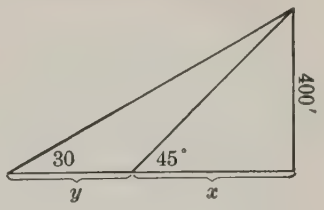
18. The distance from a point P (on level ground) to the foot of a building is 300 feet. The angle of elevation of the top of the building from P is 60°. What is the height of the building? What is the length of a straight wire from P to the top of the building? Check by use of a scale drawing.

19. Given a building 400 feet high. The angle of elevation of the top from one observer is 45° and from the other

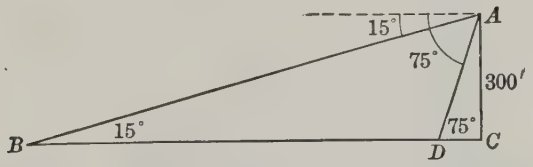
observer, who is due west (on level ground) from the first observer, it is 30°. What is the distance between the two observers?

Hint: Find x.

$$\tan 30° = \frac{400}{y + \text{value of } x}.$$



20. Two boats are in a direct westerly line from an observer who is stationed on a hill 300 feet high. The angle of depression to the first boat is 75° and the second boat is 15°. How far are the boats apart?



Hint: From a fact in geometry we know that angle B is also 15° and angle D is also 75°.

21. If the angle of depression of the river flowing through the Grand Cañon of the Colorado when observed from the brink of the Cañon at a horizontal distance of 3 miles is 17.5°, what is the depth of the Cañon?

22. By the use of the protractor and graph paper find the missing parts of the following right triangles, and then check with your table solutions.

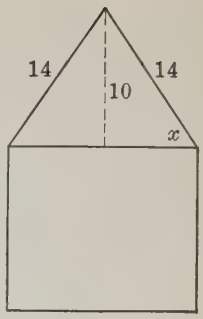
(1) $A = 45°$ $a = 60$ ft. (3) $A = 25°$ $b = 30$ ft.
(2) $B = 30°$ $b = 75$ ft. (4) $a = 40$ ft. $b = 50$ ft.

23. In the gable of my house the peak is 10 feet higher than the eaves and the rafters are 14 feet long. At what angle do the rafters slope?

See diagram. *Hint:* $\sin x = \frac{10}{14}$.

Using the same diagram, find the width of my house.

Hint: $\cos x = \frac{y}{14}$. The width $= 2y$.

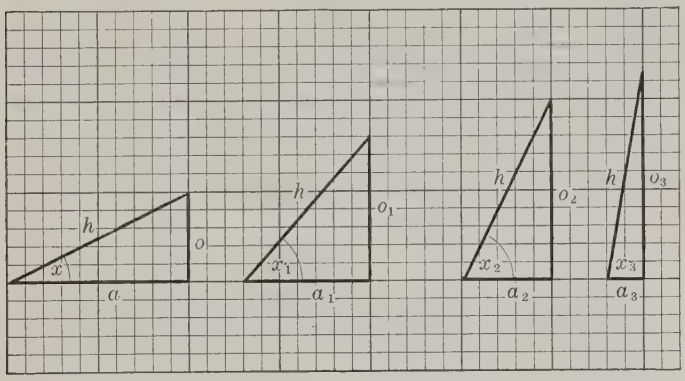


24. If the rafters of a house slope at an angle of 32 degrees and the house is 30 feet wide, how high is the ridge-pole above the eaves? How long are the rafters if they project one foot at the eaves?

THE HOUSE OF THE SEVEN GABLES

176. Dependence in Trigonometry. — You have seen how the sine, cosine, and tangent of an acute angle depend on the size of the angle. A study of the following diagrams may make the exact nature of this dependence clearer.

In each of the triangles h is the same length. As the angle x becomes larger what change takes place in the side o? In the side a?



Since $\sin x = \frac{o}{h}$, what change takes place in $\sin x$ as x is increased? Answer the same questions for $\cos x$ and for $\tan x$. Which functions increase as the angle increases? Which changes more rapidly, $\sin x$ or $\tan x$? Can you explain why?

If $\tan x = \frac{1}{50}$, is the angle small or large?

If $\cos x = \frac{1}{50}$, is the angle small or large?

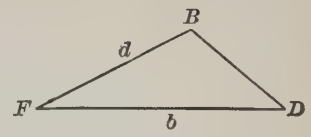
If $\sin x = \frac{9}{10}$, is the angle small or large?

If $\tan x = \frac{21}{2}$, is the angle small or large?

Draw diagrams to illustrate your answers in each case.

177. Trigonometric Functions. — The tangent, sine, and cosine of an angle are called *trigonometric functions* of the angle, because each of these ratios is a quantity which *depends* on the *size* of the angle. As the angle is changed, a corresponding change takes place in the function.

178. Trigonometric Formula Used to Find Area of a Triangle. — Of the many practical applications of trigonometry, one of the most interesting is finding the area of a triangle. We are able to prove that the area of a triangle equals one half the product of any two sides by the sine of the angle between those sides, or in the adjoining figure,

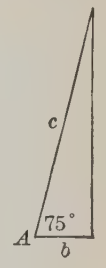


$$\text{Area} = \frac{bd \sin F}{2}.$$

Exercises

1. Suppose that BF in the preceding figure were 300 feet and FD were 400 feet and angle F were 30°. Find the area.

2. Given the angle $A = 75°$, $c = 40$, and $b = 10$. Find the area.



3. Two of the sides of a military camp were 4 miles and 6 miles, and the angle between them was 28°. Find the area of the camp.

4. A piece of land in the form of a triangle has two sides of 25 feet and 37 feet and the angle between them is 34.5 degrees. Find the area.

True-False Test

Tell whether each of the following statements is true or false:

1. The sine of an acute angle increases if the angle is made larger.

2. In the diagram on page 431, $\tan A = \frac{b}{a}$.

3. In the diagram on page 431, $\sin B = \cos A$.

4. If $\sin A = \frac{a}{c}$, $c = \frac{a}{\sin A}$.

5. $\cos A$ can never be larger than 1.

6. $\tan A$ can be larger than 1.

7. Since $\tan A = \frac{a}{b}$, $b = a \tan A$.

8. The sine of an angle is a number.

9. $\tan A \cdot \cos A = \sin A$.

10. The trigonometric functions are ratios.

11. A function is a constant.

12. The tangent of 45° is 1.

13. If the tangent of an angle is greater than 1, the angle is less than 45°.

Review

1. Write the product of $(5x + 3y)$ by $(3x - 4y)$, using the cross products method.

2. What is the height of 12 blocks of stone placed one on top of the other if each is 5 inches thick? If x inches thick? If $(x + 4)$ inches thick?

3. How many quarts of milk are there in 5 empty quart milk bottles? In 100 such bottles? In 1000? $1000 \cdot 0 = ?$

4. Four times a certain number plus ten times the result of decreasing the number by ten is equal to one hundred thirty-eight. Find the number.

5. $4x - [3x - (x + 1) - 2]$.

6. The dividend is $x^2 + 3x - 58$. The quotient is $x - 7$. The remainder is 12. Find the divisor.

7. If the length of a rectangle is 9 feet more than three times the result of the width decreased by 3 feet, what represents the length if w represents the width in feet?

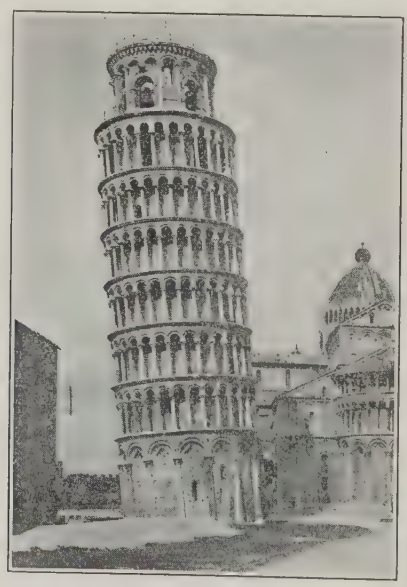
8. Add $4x^2 - 3x + 5$ and $2x^2 - 6x + 10$ and check, when $x = 3$.

9. A man has 25 coins, dimes and nickels, valued at \$2.10. how many of each has he?

10. Factor $12x^2 + 24xy - 72y^2$.

11. Divide $x^2 + 16x + 9$ by $x + 3$ and name the quotient and remainder.

12. Find the value of $\frac{y^2}{x^2} + \frac{z^2}{y^2} - \frac{u^2}{z^2}$ when $u = 5$, $x = 6$, $y = 3$, and $z = 4$.



THE LEANING TOWER OF PISA

13. If the height of the leaning tower of Pisa in Italy is 10 feet more than thirteen times the distance that it leans over, and its height is 179 feet, how many feet does it lean over?

14. Insert in a parenthesis preceded by a minus sign, the last three terms of $6x^2 - a^2 + 2ab - b^2$.

15. If $x = 2$, $y = 3$, $c = 4$, and $d = 5$, find the value of each of the following:

(a) $\dfrac{4x + 6(y + 4)}{2d - c}$.

(b) $(3x + c)(d + 3y) - (d - x)(2y - c)$.

(c) $d(c - x) - (y + d)$.

16. Find three consecutive even numbers whose sum is 108.

17. If a represents one number, and b represents another number, what does each of the following expressions represent?

$$a + b,\ a - b,\ ab,\ \frac{a}{b},\ 2a + 3b,\ a^2b,\ 5a^2b^3.$$

Solve the following equations:

18. $\dfrac{y}{3} - 12 - \dfrac{y - 3}{3} = -\dfrac{y + 4}{2} - y.$

19. $\dfrac{x - 7}{4} - \dfrac{3}{2} = \dfrac{8 - x}{6}.$

20. $\dfrac{x}{2} - 4\frac{1}{3} = \dfrac{x - 2}{3} - \dfrac{x}{5}.$

21. $.1x + 2.4 = .04x.$

22. $T = 2\pi R(R + h)$ for h.

23. The formula $d = 25t$ represents the distance that a person travels at the rate of 25 miles per hour in t hours. Construct a table giving the values of d corresponding to $t = 1, 2\ 3, 4, 5, 6$.

24. Make a graph from example 23.

25. If $\dfrac{x}{y}$ is a proper fraction how does the value of the fraction change when

(a) x is increased?

(b) x is decreased?

(c) y is increased?

(d) y is decreased?

Table of Sines, Cosines, and Tangents

° ′	Tan	Sine	Cosine	° ′	Tan	Sine	Cosine
0 00	.0000	.0000	1.0000	9 00	.1584	.1564	.9877
10	.0029	.0029	1.0000	10	.1614	.1593	.9872
20	.0058	.0058	1.0000	20	.1644	.1622	.9868
30	.0087	.0087	1.0000	30	.1673	.1650	.9863
40	.0116	.0116	.9999	40	.1703	.1679	.9858
50	.0145	.0145	.9999	50	.1733	.1708	.9853
1 00	.0175	.0175	.9998	10 00	.1763	.1736	.9848
10	.0204	.0204	.9998	10	.1793	.1765	.9843
20	.0233	.0233	.9997	20	.1823	.1794	.9838
30	.0262	.0262	.9997	30	.1853	.1822	.9833
40	.0291	.0291	.9996	40	.1883	.1851	.9827
50	.0320	.0320	.9995	50	.1914	.1880	.9822
2 00	.0349	.0349	.9994	11 00	.1944	.1908	.9816
10	.0378	.0378	.9993	10	.1974	.1937	.9811
20	.0407	.0407	.9992	20	.2004	.1965	.9805
30	.0437	.0436	.9990	30	.2035	.1994	.9799
40	.0466	.0465	.9989	40	.2065	.2022	.9793
50	.0495	.0494	.9988	50	.2095	.2051	.9787
3 00	.0524	.0523	.9986	12 00	.2126	.2079	.9781
10	.0553	.0552	.9985	10	.2156	.2108	.9775
20	.0582	.0581	.9983	20	.2186	.2136	.9769
30	.0612	.0610	.9981	30	.2217	.2164	.9763
40	.0641	.0640	.9980	40	.2247	.2193	.9757
50	.0670	.0669	.9978	50	.2278	.2221	.9750
4 00	.0699	.0698	.9976	13 00	.2309	.2250	.9744
10	.0729	.0727	.9974	10	.2339	.2278	.9737
20	.0758	.0756	.9971	20	.2370	.2306	.9730
30	.0787	.0785	.9969	30	.2401	.2334	.9724
40	.0816	.0814	.9967	40	.2432	.2363	.9717
50	.0846	.0843	.9964	50	.2462	.2391	.9710
5 00	.0875	.0872	.9962	14 00	.2493	.2419	.9703
10	.0904	.0901	.9959	10	.2524	.2447	.9696
20	.0934	.0929	.9957	20	.2555	.2476	.9689
30	.0963	.0958	.9954	30	.2586	.2504	.9681
40	.0992	.0987	.9951	40	.2617	.2532	.9674
50	.1022	.1016	.9948	50	.2648	.2560	.9667
6 00	.1051	.1045	.9945	15 00	.2679	.2588	.9659
10	.1080	.1074	.9942	10	.2711	.2616	.9652
20	.1110	.1103	.9939	20	.2742	.2644	.9644
30	.1139	.1132	.9936	30	.2773	.2672	.9636
40	.1169	.1161	.9932	40	.2805	.2700	.9628
50	.1198	.1190	.9929	50	.2836	.2728	.9621
7 00	.1228	.1219	.9925	16 00	.2867	.2756	.9613
10	.1257	.1248	.9922	10	.2899	.2784	.9605
20	.1287	.1276	.9918	20	.2931	.2812	.9596
30	.1317	.1305	.9914	30	.2962	.2840	.9588
40	.1346	.1334	.9911	40	.2994	.2868	.9580
50	.1376	.1363	.9907	50	.3026	.2896	.9572
8 00	.1405	.1392	.9903	17 00	.3057	.2924	.9563
10	.1435	.1421	.9899	10	.3089	.2952	.9555
20	.1465	.1449	.9894	20	.3121	.2979	.9546
30	.1495	.1478	.9890	30	.3153	.3007	.9537
40	.1524	.1507	.9886	40	.3185	.3035	.9528
50	.1554	.1536	.9881	50	.3217	.3062	.9520

TABLE OF SINES, COSINES, AND TANGENTS (*Continued*)

° ′	TAN	SINE	COSINE	° ′	TAN	SINE	COSINE
18 00	.3249	.3090	.9511	**27 00**	.5095	.4540	.8910
10	.3281	.3118	.9502	**10**	.5132	.4566	.8897
20	.3314	.3145	.9492	**20**	.5169	.4592	.8884
30	.3346	.3173	.9483	**30**	.5206	.4617	.8870
40	.3378	.3201	.9474	**40**	.5243	.4643	.8857
50	.3411	.3228	.9465	**50**	.5280	.4669	.8843
19 00	.3443	.3256	.9455	**28 00**	.5317	.4695	.8829
10	.3476	.3283	.9446	**10**	.5354	.4720	.8816
20	.3508	.3311	.9436	**20**	.5392	.4746	.8802
30	.3541	.3338	.9426	**30**	.5430	.4772	.8788
40	.3574	.3365	.9417	**40**	.5467	.4797	.8774
50	.3607	.3393	.9407	**50**	.5505	.4823	.8760
20 00	.3640	.3420	.9397	**29 00**	.5543	.4848	.8746
10	.3673	.3448	.9387	**10**	.5581	.4874	.8732
20	.3706	.3475	.9377	**20**	.5619	.4899	.8718
30	.3739	.3502	.9367	**30**	.5658	.4924	.8704
40	.3772	.3529	.9356	**40**	.5696	.4950	.8689
50	.3805	.3557	.9346	**50**	.5735	.4975	.8675
21 00	.3839	.3584	.9336	**30 00**	.5774	.5000	.8660
10	.3872	.3611	.9325	**10**	.5812	.5025	.8646
20	.3906	.3638	.9315	**20**	.5851	.5050	.8631
30	.3939	.3665	.9304	**30**	.5890	.5075	.8616
40	.3973	.3692	.9293	**40**	.5930	.5100	.8601
50	.4006	.3719	.9283	**50**	.5969	.5125	.8587
22 00	.4040	.3746	.9272	**31 00**	.6009	.5150	.8572
10	.4074	.3773	.9261	**10**	.6048	.5175	.8557
20	.4108	.3800	.9250	**20**	.6088	.5200	.8542
30	.4142	.3827	.9239	**30**	.6128	.5225	.8526
40	.4176	.3854	.9228	**40**	.6168	.5250	.8511
50	.4210	.3881	.9216	**50**	.6208	.5275	.8496
23 00	.4245	.3907	.9205	**32 00**	.6249	.5299	.8480
10	.4279	.3934	.9194	**10**	.6289	.5324	.8465
20	.4314	.3961	.9182	**20**	.6330	.5348	.8450
30	.4348	.3987	.9171	**30**	.6371	.5373	.8434
40	.4283	.4014	.9159	**40**	.6412	.5398	.8418
50	.4417	.4041	.9147	**50**	.6453	.5422	.8403
24 00	.4452	.4067	.9135	**33 00**	.6494	.5446	.8387
10	.4487	.4094	.9124	**10**	.6536	.5471	.8371
20	.4522	.4120	.9112	**20**	.6577	.5495	.8355
30	.4557	.4147	.9100	**30**	.6619	.5519	.8339
40	.4592	.4173	.9088	**40**	.6661	.5544	.8323
50	.4628	.4200	.9075	**50**	.6703	.5568	.8307
25 00	.4663	.4226	.9063	**34 00**	.6745	.5592	.8290
10	.4699	.4253	.9051	**10**	.6787	.5616	.8274
20	.4734	.4279	.9038	**20**	.6830	.5640	.8258
30	.4770	.4305	.9026	**30**	.6873	.5664	.8241
40	.4806	.4331	.9013	**40**	.6916	.5688	.8225
50	.4841	.4358	.9001	**50**	.6959	.5712	.8208
26 00	.4877	.4384	.8988	**35 00**	.7002	.5736	.8192
10	.4913	.4410	.8975	**10**	.7046	.5760	.8175
20	.4950	.4436	.8962	**20**	.7089	.5783	.8158
30	.4986	.4462	.8949	**30**	.7133	.5807	.8141
40	.5022	.4488	.8936	**40**	.7177	.5831	.8124
50	.5059	.4514	.8923	**50**	.7221	.5854	.8107

Table of Sines, Cosines, and Tangents (*Continued*)

° ′	Tan	Sine	Cosine	° ′	Tan	Sine	Cosine
36 00	.7265	.5878	.8090	45 00	1.0000	.7071	.7071
10	.7310	.5901	.8073	10	1.0058	.7092	.7050
20	.7355	.5925	.8056	20	1.0117	.7112	.7030
30	.7400	.5948	.8039	30	1.0176	.7133	.7009
40	.7445	.5972	.8021	40	1.0235	.7153	.6988
50	.7490	.5995	.8004	50	1.0295	.7173	.6967
37 00	.7536	.6018	.7986	46 00	1.0355	.7193	.6947
10	.7581	.6041	.7969	10	1.0416	.7214	.6926
20	.7627	.6065	.7951	20	1.0477	.7234	.6905
30	.7673	.6088	.7934	30	1.0538	.7254	.6884
40	.7720	.6111	.7916	40	1.0599	.7274	.6862
50	.7766	.6134	.7898	50	1.0661	.7294	.6841
38 00	.7813	.6157	.7880	47 00	1.0724	.7314	.6820
10	.7860	.6180	.7862	10	1.0786	.7333	.6799
20	.7907	.6202	.7844	20	1.0850	.7353	.6777
30	.7954	.6225	.7826	30	1.0913	.7373	.6756
40	.8002	.6248	.7808	40	1.0977	.7392	.6734
50	.8050	.6271	.7790	50	1.1041	.7412	.6713
39 00	.8098	.6293	.7771	48 00	1.1106	.7431	.6691
10	.8146	.6316	.7753	10	1.1171	.7451	.6670
20	.8195	.6338	.7735	20	1.1237	.7470	.6648
30	.8243	.6361	.7716	30	1.1303	.7490	.6626
40	.8292	.6383	.7698	40	1.1369	.7509	.6604
50	.8342	.6406	.7679	50	1.1436	.7528	.6583
40 00	.8391	.6428	.7660	49 00	1.1504	.7547	.6561
10	.8441	.6450	.7642	10	1.1571	.7566	.6539
20	.8491	.6472	.7623	20	1.1640	.7585	.6517
30	.8541	.6494	.7604	30	1.1708	.7604	.6494
40	.8591	.6517	.7585	40	1.1778	.7623	.6472
50	.8642	.6539	.7566	50	1.1847	.7642	.6450
41 00	.8693	.6561	.7547	50 00	1.1918	.7660	.6428
10	.8744	.6583	.7528	10	1.1988	.7679	.6406
20	.8796	.6604	.7509	20	1.2059	.7698	.6383
30	.8847	.6626	.7490	30	1.2131	.7716	.6361
40	.8899	.6648	.7470	40	1.2203	.7735	.6338
50	.8952	.6670	.7451	50	1.2276	.7753	.6316
42 00	.9004	.6691	.7431	51 00	1.2349	.7771	.6293
10	.9057	.6713	.7412	10	1.2423	.7790	.6271
20	.9110	.6734	.7392	20	1.2497	.7808	.6248
30	.9163	.6756	.7373	30	1.2572	.7826	.6225
40	.9217	.6777	.7353	40	1.2647	.7844	.6202
50	.9271	.6799	.7333	50	1.2723	.7862	.6180
43 00	.9325	.6820	.7314	52 00	1.2799	.7880	.6157
10	.9380	.6841	.7294	10	1.2876	.7898	.6134
20	.9435	.6862	.7274	20	1.2954	.7916	.6111
30	.9490	.6884	.7254	30	1.3032	.7934	.6088
40	.9545	.6905	.7234	40	1.3111	.7951	.6065
50	.9601	.6926	.7214	50	1.3190	.7969	.6041
44 00	.9657	.6947	.7193	53 00	1.3270	.7986	.6018
10	.9713	.6967	.7173	10	1.3351	.8004	.5995
20	.9770	.6988	.7153	20	1.3432	.8021	.5972
30	.9827	.7009	.7133	30	1.3514	.8039	.5948
40	.9884	.7030	.7112	40	1.3597	.8056	.5925
50	.9942	.7050	.7092	50	1.3680	.8073	.5901

TABLE OF SINES, COSINES, AND TANGENTS (*Continued*)

° ′	TAN	SINE	COSINE	° ′	TAN	SINE	COSINE
54 00	1.3764	.8090	.5878	**63 00**	1.9626	.8910	.4540
10	1.3848	.8107	.5854	**10**	1.9768	.8923	.4514
20	1.3934	.8124	.5831	**20**	1.9912	.8936	.4488
30	1.4019	.8141	.5807	**30**	2.0057	.8949	.4462
40	1.4106	.8158	.5783	**40**	2.0204	.8962	.4436
50	1.4193	.8175	.5760	**50**	2.0353	.8975	.4410
55 00	1.4281	.8192	.5736	**64 00**	2.0503	.8988	.4384
10	1.4370	.8208	.5712	**10**	2.0655	.9001	.4358
20	1.4460	.8225	.5688	**20**	2.0809	.9013	.4331
30	1.4550	.8241	.5664	**30**	2.0965	.9026	.4305
40	1.4641	.8258	.5640	**40**	2.1123	.9038	.4279
50	1.4733	.8274	.5616	**50**	2.1283	.9051	.4253
56 00	1.4826	.8290	.5592	**65 00**	2.1445	.9063	.4226
10	1.4919	.8307	.5568	**10**	2.1609	.9075	.4200
20	1.5013	.8323	.5544	**20**	2.1775	.9088	.4173
30	1.5108	.8339	.5519	**30**	2.1943	.9100	.4147
40	1.5204	.8355	.5495	**40**	2.2113	.9112	.4120
50	1.5301	.8371	.5471	**50**	2.2286	.9124	.4094
57 00	1.5399	.8387	.5446	**66 00**	2.2460	.9135	.4067
10	1.5497	.8403	.5422	**10**	2.2637	.9147	.4041
20	1.5597	.8418	.5398	**20**	2.2817	.9159	.4014
30	1.5697	.8434	.5373	**30**	2.2998	.9171	.3987
40	1.5798	.8450	.5348	**40**	2.3183	.9182	.3961
50	1.5900	.8465	.5324	**50**	2.3369	.9194	.3934
58 00	1.6003	.8480	.5299	**67 00**	2.3559	.9205	.3907
10	1.6107	.8496	.5275	**10**	2.3750	.9216	.3881
20	1.6212	.8511	.5250	**20**	2.3945	.9228	.3854
30	1.6319	.8526	.5225	**30**	2.4142	.9239	.3827
40	1.6426	.8542	.5200	**40**	2.4342	.9250	.3800
50	1.6534	.8557	.5175	**50**	2.4545	.9261	.3773
59 00	1.6643	.8572	.5150	**68 00**	2.4751	.9272	.3746
10	1.6753	.8587	.5125	**10**	2.4960	.9283	.3719
20	1.6864	.8601	.5100	**20**	2.5172	.9293	.3692
30	1.6977	.8616	.5075	**30**	2.5386	.9304	.3665
40	1.7090	.8631	.5050	**40**	2.5605	.9315	.3638
50	1.7205	.8646	.5025	**50**	2.5826	.9325	.3611
60 00	1.7321	.8660	.5000	**69 00**	2.6051	.9336	.3584
10	1.7437	.8675	.4975	**10**	2.6279	.9346	.3557
20	1.7556	.8689	.4950	**20**	2.6511	.9356	.3529
30	1.7675	.8704	.4924	**30**	2.6746	.9367	.3502
40	1.7796	.8718	.4899	**40**	2.6985	.9377	.3475
50	1.7917	.8732	.4874	**50**	2.7228	.9387	.3448
61 00	1.8040	.8746	.4848	**70 00**	2.7475	.9397	.3420
10	1.8165	.8760	.4823	**10**	2.7725	.9407	.3393
20	1.8291	.8774	.4797	**20**	2.7980	.9417	.3365
30	1.8418	.8788	.4772	**30**	2.8239	.9426	.3338
40	1.8546	.8802	.4746	**40**	2.8502	.9436	.3311
50	1.8676	.8816	.4720	**50**	2.8770	.9446	.3283
62 00	1.8807	.8829	.4695	**71 00**	2.9042	.9455	.3256
10	1.8940	.8843	.4669	**10**	2.9319	.9465	.3228
20	1.9074	.8857	.4643	**20**	2.9600	.9474	.3201
30	1.9210	.8870	.4617	**30**	2.9887	.9483	.3173
40	1.9347	.8884	.4592	**40**	3.0178	.9492	.3145
50	1.9486	.8897	.4566	**50**	3.0475	.9502	.3118

TABLE OF SINES, COSINES, AND TANGENTS (*Continued*)

° ′	TAN	SINE	COSINE	° ′	TAN	SINE	COSINE
72 00	3.0777	.9511	.3090	81 00	6.3138	.9877	.1564
10	3.1084	.9520	.3062	10	6.4348	.9881	.1536
20	3.1397	.9528	.3035	20	6.5606	.9886	.1507
30	3.1716	.9537	.3007	30	6.6912	.9890	.1478
40	3.2041	.9546	.2979	40	6.8269	.9894	.1449
50	3.2371	.9555	.2952	50	6.9682	.9899	.1421
73 00	3.2709	.9563	.2924	82 00	7.1154	.9903	.1392
10	3.3052	.9572	.2896	10	7.2687	.9907	.1363
20	3.3402	.9580	.2868	20	7.4287	.9911	.1334
30	3.3759	.9588	.2840	30	7.5958	.9914	.1305
40	3.4124	.9596	.2812	40	7.7704	.9918	.1276
50	3.4495	.9605	.2784	50	7.9530	.9922	.1248
74 00	3.4874	.9613	.2756	83 00	8.1443	.9925	.1219
10	3.5261	.9621	.2728	10	8.3450	.9929	.1190
20	3.5656	.9628	.2700	20	8.5555	.9932	.1161
30	3.6059	.9636	.2672	30	8.7769	.9936	.1132
40	3.6470	.9644	.2644	40	9.0098	.9939	.1103
50	3.6891	.9652	.2616	50	9.2553	.9942	.1074
75 00	3.7321	.9659	.2588	84 00	9.5144	.9945	.1045
10	3.7760	.9667	.2560	10	9.7882	.9948	.1016
20	3.8208	.9674	.2532	20	10.078	.9951	.0987
30	3.8667	.9681	.2504	30	10.385	.9954	.0958
40	3.9136	.9689	.2476	40	10.712	.9957	.0929
50	3.9617	.9696	.2447	50	11.059	.9959	.0901
76 00	4.0108	.9703	.2419	85 00	11.430	.9962	.0872
10	4.0611	.9710	.2391	10	11.826	.9964	.0843
20	4.1126	.9717	.2363	20	12.251	.9967	.0814
30	4.1653	.9724	.2334	30	12.706	.9969	.0785
40	4.2193	.9730	.2306	40	13.197	.9971	.0756
50	4.2747	.9737	.2278	50	13.727	.9974	.0727
77 00	4.3315	.9744	.2250	86 00	14.301	.9976	.0698
10	4.3897	.9750	.2221	10	14.924	.9978	.0669
20	4.4494	.9757	.2193	20	15.605	.9980	.0640
30	4.5107	.9763	.2164	30	16.350	.9981	.0610
40	4.5736	.9769	.2136	40	17.169	.9983	.0581
50	4.6382	.9775	.2108	50	18.075	.9985	.0552
78 00	4.7046	.9781	.2079	87 00	19.081	.9986	.0523
10	4.7729	.9787	.2051	10	20.206	.9988	.0494
20	4.8430	.9793	.2022	20	21.470	.9989	.0465
30	4.9152	.9799	.1994	30	22.904	.9990	.0436
40	4.9894	.9805	.1965	40	24.542	.9992	.0407
50	5.0658	.9811	.1937	50	26.432	.9993	.0378
79 00	5.1446	.9816	.1908	88 00	28.636	.9994	.0349
10	5.2257	.9822	.1880	10	31.242	.9995	.0320
20	5.3093	.9827	.1851	20	34.368	.9996	.0291
30	5.3955	.9833	.1822	30	38.188	.9997	.0262
40	5.4845	.9838	.1794	40	42.964	.9997	.0233
50	5.5764	.9843	.1765	50	49.104	.9998	.0204
80 00	5.6713	.9848	.1736	89 00	57.290	.9998	.0175
10	5.7694	.9853	.1708	10	68.750	.9999	.0145
20	5.8708	.9858	.1679	20	85.940	.9999	.0116
30	5.9758	.9863	.1650	30	114.59	1.0000	.0087
40	6.0844	.9868	.1622	40	171.89	1.0000	.0058
50	6.1970	.9872	.1593	50	343.77	1.0000	.0029
				90 00	infinite	1.0000	.0000

SUPPLEMENTARY EXERCISES

Evaluating Formulas

Find the value of the required letter and evaluate.

1. $F = 32 + 1.8\,C$ if $C = 100°$. If $C = 0°$. For F.
2. $A = P(1 + rt)$ for P if $A = \$1040$, $t = 2$, and $r = 6\%$.
3. $S = 6\,e^2$ (surface of a cube) for S, if $e = 2$. For e if $S = 96$.
4. $A = lw$ for w when $A = 1500$ and $l = 60$.
5. $I = prt$ for I, when $p = \$240$, $r = 6\%$, and $t = 4$.
6. $P = 2\,l + 2\,w$ for P, when $l = 20$ and $w = 15$.
7. $A = 4\,\pi r^2$ for A, if $r = 7$.
8. $C = \pi d$ for C, given $\pi = \frac{22}{7}$, and $d = 42$.
9. $A = \frac{1}{4}\,\pi d^2$ for A, if $\pi = \frac{22}{7}$ and $d = 14$.
10. $F = \dfrac{mv}{t}$ for F, if $m = 420$, $t = 30$, and $v = 61$.
11. $K = Wh$ for K, if $W = 800$ lb. and $h = 12$ ft.
12. $P = \dfrac{WH}{L}$ for P, if $W = 7000$ lb., $L = 640$ ft., and $H = 85$ ft.
13. $f = \dfrac{P}{W}$ for P, if $f = .1666$ and $W = 10{,}000$ lb.
14. $L = 2\,l + \dfrac{(D + d)\pi}{2}$ for L, if $D = 5$ ft., $d = 2$ ft., and $l = 12$ ft.
15. R. P. M. $= \dfrac{80}{D}$ (a steel drill) for R. P. M., if D (diam. of drill) $= \frac{3}{4}$ inch.
16. $F = \dfrac{WH}{D}$ (hammer blow) for F, if $W = 400$ lb., $H = 4$ ft., and $D = .002804$.

17. $S = 16\,t^2$ for t, if $S = 240$ feet.

18. $C = \dfrac{E}{R}$ for E, if $C = 10$ and $R = 5$.

19. $v = \dfrac{HB}{3}$ for B, if $v = 216$ and $H = 6$.

20. $A = \frac{1}{2}\,bh$ for h, if $A = 864$ and $b = 48$.

21. $V = \frac{1}{3}\,bh$ for b, if $h = 17$ and $V = 238$.

22. $S = \frac{1}{2}\,gt^2$ for g, if $S = 402.5$ and $t = 5$.

23. $V = lwh$ for l, if $V = 10{,}200$, $w = 17$, and $h = 15$.

24. $K = \frac{1}{2}\,Mv^2$ for K, if $v = 50$ and $M = 300$. (Physics formula.)

25. $V = \frac{1}{3}\,\pi r^2 h$ for r, if $V = 528$, $\pi = \frac{22}{7}$, and $h = 14$.

26. Find the area of a parallelogram whose base is 14 inches, and whose altitude is 6 inches.

27. Find the side or lateral area of a cylinder ($S = 2\,\pi rh$) when $\pi = \frac{22}{7}$, $r = 14''$, and $h = 6''$.

28. Find the distance a ball will fall from rest ($S = \frac{1}{2}\,gt^2$) when $g = 32.16$ and $t = 5$ seconds.

29. Find the wages from $w = \$.40\,h$ when $h = 6\frac{1}{2}$ hours.

30. Find the cost from $c = \$.30\,d$ if $d = 7$ dozen oranges.

31. Find the weight of water from $w = 62.5\,v$ if $v = 17$ cubic feet.

32. Find the specific gravity from s.g. $= \dfrac{w_b}{w_w}$ where w_b (weight of the body) is 556.25 pounds and w_w (weight of an equal volume of water) is 62.5 pounds.

33. If $K = \frac{1}{2}\,Mv^2$ (a physics formula), find K when $M = 400$ and $v = 25$.

34. What is the area of a circle whose radius is $28''$?

35. What is the volume of a cylinder ($V = \pi r^2 h$) if $\pi = \frac{22}{7}$, $r = 7''$, and $h = 14''$?

Addition

Add. Check the first 10 examples:

1. $3x - 6y$, $2x - 3y$, $4x + 7y$, $-3x + 8y$.

2. $-8x + 2y$, $4x - 5y$, $9x - 7y$, $-2x - 9y$.

3. $5xy + 7x$, $4xy - 9x$, $-10xy - 8x$.

4. $2x + 3y - 5z$, $-3x - 6y + 4z$, $7x - 3y + 2z$.

5. $-5a + 4b - c$, $7a + 7b - 3c$, $3a - 5b - 4c$.

6. $6x + 5y + 7z$, $4x + 3y + 6z$, and $8x - 5y + 9z$.

7. $3x - 7y - 5z$, $4x + 8y - 2z$, and $4x - 6y - 3z$.

8. $5K + 3H$, $-7K - 9H$, $-2K + 3H$, and $5K + 4H$.

9. $-4x+10y-z$, $5x-9y+2z$, and $-7x+8y-5z$.

10. $8x^2 + 7x + 9$, $5x^2 - 8x - 8$, and $-7x^2 - 3x - 4$.

11. $7x^2-9xy+y$, $-5x^2+4xy-2y$, and $3x^2-8xy+7y$.

12. $-8x+7y^2+3z$, $5x-5y^2-4z$, and $4x-7y^2+2z$.

13. $6ax + 4by - 3cz$, $2ax - 6by - 2cz$, and $-8ax - 7by - 11cz$.

14. $3(x + y)$, $-7(x + y)$, $9(x + y)$, and $-2(x + y)$.

15. $8(x + y) - 3(w - z)$, $-7(x + y) - 4(w - z)$, and $9(x + y) + 8(w - z)$.

16. $9(x - y) + 4(x - z)$, $-7(x - y) - 5(x - z)$, and $-8(x - y) - 8(x - z)$.

17. $5(x + 4) + 6(x - 1) + 7(x + 3)$, $7(x + 4) - 4(x + 3) - 5(x - 1)$, and $3(x + 4) - 8(x - 1) + (x + 4)$.

18. $\frac{1}{2}x + \frac{1}{3}y - \frac{1}{4}z$ and $\frac{1}{2}x + \frac{2}{3}y - \frac{3}{4}z$.

19. $\frac{1}{2}x^2 + \frac{1}{3}x + \frac{5}{6}$, $\frac{1}{4}x^2 - \frac{1}{6}x + \frac{3}{4}$, and $\frac{2}{3}x^2 - \frac{5}{6}x - \frac{1}{12}$.

20. $.7a + .3b - .6c$ and $.2a - .6b + .3c$.

21. $.12x^3 + .15x^2 + 2.3x + 3.4$, $.36x^3 - .27x^2 - 3.5x - 2.6$ and $.2x^3 + .3x^2 + .5x$.

Subtraction

1. From $3x - 5y$ subtract $x + 3y$.
2. From $9x + 7y$ subtract $-3x + 5y$.
3. From $5x - 6y$ subtract $9x + 8y$.
4. Subtract $9a - 10b$ from $6a - 8b$.
5. Take $7K - 11H$ from $4K - 9H$.
6. From $3\frac{1}{4}x - 4y$ subtract $5x + 7y$.
7. From $2x - 3y - 4z$ subtract $6x - 7y - 4z$.
8. Subtract $9x^2 + 4xy$ from $8x^2 - 5xy$.
9. Subtract $7H - 9K - 7M$ from $5H + 6K + M$.
10. From $8x^3 - 6x^2 + 5x - 7$ subtract $9x^3 - 7x^2 - 6x - 11$.
11. From $6x^3 - 7x^2 + 3x - 18$ take $8x^3 + 7x^2 - 4x$.
12. Subtract $3x^2 - 5x + 6$ from $8x^4 - 9x^3 + 7x^2 - 16$.
13. What must be added to $7x^3 + 5x^2 - 4x - 3$ to give $8x^3 + 5x^2 - 4x - 7$?
14. From $3x^2 + 5xy - y^2$ subtract $6x^2 + 5xy - 4y^2$.
15. Subtract $-17x^2 - 42xy + 17y^2$ from $25x^2 - 36xy$.
16. From $125x^3 - 16x^2 + 87x - 96$ subtract $-15x^3 - 18x^2 + 95x - 75$.
17. Subtract $-23x^2 - 100x - 65$ from $-19x^2 + 42x - 63$.
18. From $-5x^3 - 3x^2 + 4$ subtract $2x^3 - 3x^2 - 5x + 8$.
19. From $6x^2 + 8x - 2$ take $2x^2 - 3x + 5$ and check, letting $x = 2$.
20. From $7.3x^3 + 6.7x^2 - 5.4x - 8.7$ subtract $4.5x^3 + 5.9x^2 + 3.5x - 9.6$.
21. Subtract $2.5x^3 - .56x^2 + .25x + .75$ from $3.5x^3 - .65x^2 - .64x + 1.25$.
22. From $\frac{1}{2}H + \frac{1}{3}K - \frac{5}{8}$ take $\frac{2}{3}H - \frac{3}{4}K + \frac{2}{3}$.
23. What is the remainder when $(a + b)x + (c + d)y + (h + k)z$ is subtracted from $2(a+b)x - 3(c+d)y - 4(h+k)z$?

Multiplication

Multiply:

1. $x^3 - x - 2$ by $2x^3 + 3x + 4$. Check.
2. $3x^2 + 2xy - 8y^2$ by $2x^2 - 5xy + 4y^2$.
3. $.4x^2 + .5x - 2.6$ by $.35x^2 - .63x + .75$.
4. $.6x^2 - .5x + 3.5$ by $.25x^2 + .55x - .35$.
5. $x^2 - bx - 1$ by $x + y$.
6. $4H^2 - 5K^2 - 6H + 7K$ by $H + K$.
7. $\frac{1}{2}x^3 + \frac{1}{3}x^2 + \frac{1}{4}x - \frac{1}{5}$ by $\frac{1}{3}x^2 - \frac{1}{2}x + \frac{1}{4}$.
8. $x^{2a} - 3x^a - 5$ by $x^a - 4$.
9. $x^n + 2x^{n-1} + x^{n-2}$ by $3x - 2$.
10. $3(a+b)+5(c+d)-6(h+k)$ by $4(a+b)-5(c+d)$.

Division

Divide:

1. $54x^3 + 18x^2 - 6x - 20$ by $3x - 2$.
2. $58x + 7x^3 - 21 - 24x^2$ by $7x - 3$.
3. $23x^2 - 48 + 6x^4 - 2x - 31x^3$ by $3x^2 + 6 - 5x$.
4. $6x^3 + 5x^2 - 4x + 3$ by $2x + 3$.
5. $11x^3 + x^4 + 23x^2 - 55x - 140$ by $x^2 - 5$.
6. $6m - 9m^2 + 4m^4 - 1$ by $3m + 2m^2 - 1$.
7. $m^2 + 9 + 6m - 25x^2$ by $3 + m - 5x$.
8. $15t^4 - 19t^3 + 8t^2 - t - 1$ by $5t^2 - 3t - 1$.
9. $6e^2 - 13e + 6$ by $2e - 3$ and check with $e = 2$.
10. $49x^4 - 25x^2 - 20x - 24$ by $7x^2 - 5x - 4$.
11. $a^2 + 3ab - 10b^2$ by $a - 2b$.
12. $.08H^3 - .04H^2K + .04HK^2 - .08K^3$ by $.2H - .2K$.
13. $\frac{9}{16}x^4 - \frac{7}{4}x^2 - \frac{3}{4}x^3 + \frac{4}{3}x + \frac{16}{9}$ by $\frac{3}{2}x^2 - x - \frac{8}{3}$.
14. $6x^{3n} - 25x^{2n} + 17x^n + 20$ by $2x^n - 5$.
15. $(a+b)^2+2(a+b)(a-b)+(a-b)^2$ by $(a+b)+(a-b)$.

Fractional Equations

Solve and check as directed:

1. $\frac{1}{2}(x-3)-\frac{1}{3}(x-6)=2.$
2. $\frac{1}{4}(7x+5)-\frac{1}{5}(6x+4)=\frac{1}{7}(9x-2).$
3. $\frac{1}{8}(3x-1)=\frac{1}{4}(15-x)-2.$
4. $\frac{1}{9}(2x-1)+5=\frac{1}{3}(4x-2).$
5. $\frac{1}{2}(27-2x)=\frac{9}{2}-\frac{1}{10}(7x-54).$
6. $\frac{3}{4}(9x+2)-\frac{4}{5}(7x-15)=17.$
7. $\frac{1}{2}(x+3)-\frac{3}{5}(x+1)=2.$
8. $2x-1-\frac{1}{2}(2x-2)=\frac{1}{5}(3x+1)+\frac{1}{4}(x+1).$
9. $\dfrac{x+3}{2}-\dfrac{x-2}{3}=\dfrac{3x-5}{12}+\dfrac{1}{4}.$
10. $\dfrac{x}{2}-2\left(\dfrac{4x}{5}-3\right)=4-\dfrac{3}{2}\left(\dfrac{x}{2}+1\right).$
11. $3x-\left(2x-\dfrac{x+7}{5}\right)=\dfrac{x+4}{6}+9.$
12. $\dfrac{x+5}{4}-\dfrac{2x+4}{9}=1.$
13. $\dfrac{x-2}{5}-\dfrac{x-3}{4}=\dfrac{x-7}{10}.$
14. $\dfrac{31}{2}-\dfrac{9-5x}{6}=\dfrac{5x+3}{8}-\dfrac{3-4x}{3}+\dfrac{x}{2}.$
15. $\dfrac{3x-8}{5}-\dfrac{5x+6}{15}=\dfrac{x-3}{4}-\dfrac{2x-5}{6}-\dfrac{41}{60}.$
16. $3x-11=\dfrac{3\frac{1}{3}x+18\frac{1}{3}}{2\frac{1}{2}}.$
17. $\dfrac{3x+7}{4x+8}=\dfrac{6x-2}{8x-5}.$
18. $\dfrac{8x+20}{4x+13}=\dfrac{5x+14}{3x+11}.$
19. $\dfrac{3x-2}{2x+3}=\dfrac{5x-8}{4x-3}.$
20. $\dfrac{4x}{x+1}-\dfrac{x}{x-2}=3.$
21. $\dfrac{4x-1}{x+1}-\dfrac{3x-1}{x-1}=1.$

Factoring

Factor:

1. $4x^2 - 81$.
2. $9x^2 - y^2$.
3. $x^2 - 25y^2z^2$.
4. $x^4 - x^2$.
5. $-x^3 + x^2 - x$.
6. $8 - 18m^2$.
7. $r^2s^2 - 4r^6s^2$.
8. $25a^2b^4 - 16a^4b^2$.
9. $2a^3 - 2ay^4$.
10. $a^3b^2 - 81a^3$.
11. $2Pr^2 + 2Prh$.
12. $100x^4 - 49y^6$.
13. $\frac{1}{36}x^2 - \frac{16}{81}y^2$.
14. $x^2 - x - 12$.
15. $1 + 6ab + 9a^2b^2$.
16. $C^2 + 3C - 54$.
17. $a^2 - a - 20$.
18. $6a^2 + 7a + 2$.
19. $m^2 - m - 42$.
20. $4s^2 - 20sw + 25w^2$.
21. $x^2 - x - 132$.
22. $a^2 - 10a + 16$.
23. $2a^2 - ab - b^2$.
24. $m^2 - 11m + 18$.
25. $2a^2 + 5ab - 3b^2$.
26. $6p^2 - pq - q^2$.
27. $C^2 - C - 6$.
28. $5x^2 - 7xy + 2y^2$.
29. $4a^2 + 12ab + 9b^2$.
30. $x^2 + 2x - 24$.
31. $4a^2 - 15ab - 4b^2$.
32. $4x^2 + 24x + 36$.
33. $12 + 16c - 3c^2$.
34. $a^2 - 2a - 48$.
35. $36m^2 + 60mn + 25n^2$.
36. $x^4 + 8x^2 + 16$.
37. $10x^2 + 23x + 12$.
38. $C + Cn - 2C$.
39. $30x^3 + 82x^2 + 56x$.
40. $3x^2 + 24x - 99$.
41. $6x^2 + 3x - 45$.
42. $2x^2 + 20x + 50$.
43. $3x^2 + 5x + 2$.
44. $56x^2 - 107x + 45$.
45. $30x - 37x^2 - 77x^3$.
46. $ax^2 - 6axy + 9ay^2$.
47. $-1 - 6x - 9x^2$.
48. $-H - 2 + 3H^2$.
49. $12c^2 + 7c - 12$.
50. $ax^2 - 3ax - 70a$.
51. $2x^2 + 15xy + 7y^2$.
52. $x^2y - 25y^3$.
53. $28x^2y^2 - 7y^4$.
54. $(x+1)^2 - 10(x+1) + 25$.
55. $(a+x)^2 + 6(a+x) + 9$.
56. $1 - (x-y)^2$.
57. $4(a+b)^2 - 9(b+c)^2$.
58. $x^2 + \frac{5}{6}x + \frac{1}{6}$.

Factoring. Decimals

Factor:

1. $x^2 - .25$.
2. $.16\,x^2 - .49\,y^2$.
3. $x^2 - 2.25\,y^2$.
4. $.16\,x^2 - .09\,y^2$.
5. $x^2 + 1.8\,x + .81$.
6. $x^2 - .8\,x + .16$.
7. $.12\,x^2 - .24\,x + .12$.
8. $.09\,y^2 - .0016\,z^2$.
9. $x^2 - .9\,x + .20$.
10. $x^2 - 1.4\,x + .49$.
11. $8.6\,y^2 + 8.6\,z^2$.
12. $.25\,x^2 - \frac{1}{16}\,y^2$.
13. $.81\,x^2 - 1.69\,y^2$.
14. $.4\,x^2 - 1.6\,x + 1.6$.
15. $.7\,x^2 - 1.4\,x + .7$.
16. $.8\,y^5 - .8\,y$.
17. $6 - 2.6\,x + .24\,x^2$.
18. $.02\,x^2 - 7.02\,x + 7$.
19. $.09\,x^2 + .24\,xy + .16\,y^2$.
20. $.49\,x^2 + 1.12\,xy + .64\,y^2$.
21. $.25\,x^2 - .80\,x + .64$.
22. $49\,H^2 - 12.6\,H + .81$.
23. $.3\,x^2 - 7.5\,y^2$.
24. $H^2 - .3\,H - .28$.
25. $1.21\,H^2 - .64\,K^2$.
26. $H^2 - 2.2\,H + .4$.
27. $.7\,x^2 - 2.1\,x + 1.4$.
28. $12.25\,H^4 - .64\,K^4$.
29. $1.44\,H^2 - 1.2\,HK + .25\,K^2$.
30. $.003\,H^{2m} - .048\,K^{2n}$.

Optional

Factoring. Literal Exponents

Factor and check:

1. $x^{2a} - y^{2b}$.
2. $x^{4a} - 36$.
3. $x^{6a} - 9$.
4. $3\,x^a + 5\,x^a y$.
5. $5\,x^a + 15\,x^a y^2$.
6. $2\,x^{2a} - 32\,y^{2b}$.
7. $3\,x^{4a} - 108$.
8. $4\,x^{2m} - 9\,y^{4n}$.
9. $6\,x^{2a} - 54$.
10. $49\,x^{2a} - 36\,y^{2b}$.
11. $x^{2a} + 2\,x^a + 1$.
12. $x^{2a} + 7\,x^a + 12$.
13. $x^{4a} + 9\,x^{2a} + 20$.
14. $x^{4a} - 9\,x^{2a} + 20$.
15. $x^{2a} + 11\,x^a + 18$.
16. $25 - 10\,x^a + x^{2a}$.
17. $10\,x^{2a} + 30\,x^a - 100$.
18. $6\,x^{2a} - 7\,x^a y^b - 20\,y^{2b}$.
19. $x^{8a} - 54 + 3\,x^{4a}$.
20. $\frac{1}{4}\,H^{2x} - \frac{1}{9}\,K^{2y}$.

Find the sum, the difference, the product, and the quotient of the two fractions:

1. $\frac{4}{5}, \frac{2}{3}$.
2. $\frac{2y}{7}, \frac{5y}{6}$.
3. $\frac{3HK}{4}, \frac{4HK}{7}$.
4. $\frac{2}{3x}, \frac{1}{2x}$.
5. $\frac{5}{3x^2}, \frac{3}{5x^2}$.
6. $\frac{3x}{4y}, \frac{8y}{3x}$.
7. $\frac{2}{x-y}, \frac{3}{x+y}$.
8. $\frac{x-y}{10}, \frac{x+y}{8}$.
9. $\frac{x-y}{x}, \frac{x+y}{y}$.
10. $\frac{a-b}{a^2-b^2}, \frac{a+b}{a^2-b^2}$.

Simplify by performing the indicated operations:

11. $\frac{2x}{3} + \frac{5x}{6} - \frac{3x}{2}$.
12. $\frac{15}{2y} - \frac{9}{10y} + \frac{11}{5y}$.
13. $\frac{2x}{9} + \frac{3x}{8} - \frac{x}{12}$.
14. $\frac{y}{9} - \frac{4y}{15} + \frac{5y}{18}$.
15. $\frac{y^2}{2} + \frac{y^2}{1} - \frac{8y^2}{7}$.
16. $\frac{2x}{y^2} + \frac{3y}{x^2} + \frac{2y}{x^2y^2}$.
17. $\frac{9x}{2} \div x\left(6 - \frac{3y}{14}\right)$.
18. $\frac{5x}{3} \div x\left(5 - \frac{5y}{9}\right)$.
19. $x + 1 + \frac{2}{x+1}$.
20. $x + 1 - \frac{2}{x+1}$.
21. $x + 1 - \frac{1}{x+1}$.
22. $x + 1 + \frac{1}{x-1}$.
23. $\frac{x+3}{x-1} \cdot \frac{x^2-1}{x^2-9}$.
24. $\frac{x+y}{x-y} \div \frac{x^2+y^2}{x^2-y^2}$.
25. $\frac{3}{5x-2} - 3x + 2$.
26. $\frac{b^2}{b^2-1} + \frac{b}{b+1} - \frac{b}{1-b}$.
27. $\frac{4x}{x^2-4} + \frac{3x-2}{4-x^2}$.

28. $\dfrac{x-y}{3x+2y}+\dfrac{12xy}{9x^2-4y^2}-\dfrac{x+y}{3x-2y}$.

29. $\dfrac{a}{a-2}-\dfrac{a-2}{a+2}-\dfrac{8}{a^2-4}$.

30. $\dfrac{4}{x-1}-\dfrac{3}{x-3}+\dfrac{x-15}{x^2-4x+3}$.

31. $\dfrac{4}{a-3}-\dfrac{a-1}{a+3}+\dfrac{a^2-38a-3}{a^2-9}$.

32. $\dfrac{x+y}{x-y}+\dfrac{z+w}{z-w}+\dfrac{2(xz-yw)}{(y-x)(z-w)}$.

33. $\left(4-\dfrac{6}{a+1}\right)\div\left(8-\dfrac{4a-8}{a^2-1}\right)$.

34. $\left(\dfrac{x}{y}+1\right)\left(\dfrac{y}{x}-1\right)\left(\dfrac{y^2}{y^2-x^2}\right)$.

35. $\left(6y-11-\dfrac{7}{y}\right)\div\left(2+\dfrac{11}{y}+\dfrac{5}{y^2}\right)\div\left(\dfrac{1}{3y^3-75y}\right)$.

36. $\dfrac{x^2-y^2}{xz+z^2}\div\dfrac{xy+y^2}{x^2-z^2}\left(x+\dfrac{xy}{x-y}\right)$.

37. $\left(1-\dfrac{ab}{a^2-ab+b^2}\right)\left(1-\dfrac{3ab}{a^2+2ab+b^2}\right)\div\dfrac{(a-b)^2}{(a+b)^2}$.

38. $\dfrac{2a^2-13a+15}{4a^2-9}\times\dfrac{2a+1}{2a-1}\div\dfrac{a-5}{2a-1}$.

39. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}$.

40. $\dfrac{x^2-x-20}{x^2-25}\cdot\dfrac{x^2-x-2}{x^2+2x-8}\div\dfrac{x+1}{x^2+5x}$.

41. $\dfrac{a^2+4ab+3b^2}{a^2-b^2}\times\dfrac{(a-b)^2}{a+3b}\div\dfrac{a-b}{a+b}$.

42. $\dfrac{a(2a-b)^2}{2a+b}\times\dfrac{b}{2a^2-ab}\div\dfrac{b(4a^2-b^2)}{(2a+b)^2}$.

43. $\left(\dfrac{6a}{a^2-4}+\dfrac{3}{2-a}\right)\div\dfrac{3}{a^2-a-6}$.

44. $\dfrac{\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}}{x+\dfrac{y^2}{x}}$.

45. $\dfrac{\dfrac{x}{1+x}+\dfrac{1-x}{x}}{\dfrac{x}{1+x}-\dfrac{1-x}{x}}$.

46. $\dfrac{x}{1+x}-\dfrac{x}{1-x}+\dfrac{x^2}{x^2-1}$.

47. $\dfrac{x^2-7x+12}{x-1}\div\dfrac{x^2-16}{1-x}$.

Literal Equations

Solve for x:

1. $\dfrac{x+6b}{3}-\dfrac{x+12b}{6}=1$.

2. $\dfrac{x}{3}-\dfrac{2x-3}{4}=2c$.

3. $\dfrac{4x-8a}{4}+6a=\dfrac{6x+32a}{8}$.

4. $\dfrac{2x+14b}{7}-\dfrac{21b}{2}=-\dfrac{21b-2x}{14}$.

5. $\dfrac{x}{2b}-\dfrac{x}{3b}=3$.

6. $\dfrac{x}{a}+\dfrac{x}{2a}=3c$.

7. $\dfrac{5}{x-a}=2$.

8. $\dfrac{5}{x}-a=b$.

9. $\dfrac{5}{x-b}=\dfrac{7}{b}$.

10. $\dfrac{5}{x-b}=\dfrac{7}{x}$.

11. $\dfrac{3-x}{5-x}=\dfrac{H}{K}$.

12. $\dfrac{x-1}{x+1}=\dfrac{m+n}{m-n}$.

13. $\dfrac{H+K}{1-x}=\dfrac{H-K}{1+x}$.

14. $\dfrac{x-a}{x}=\dfrac{x}{x+b}$.

15. $\dfrac{ax}{2b}-4b^2=\dfrac{2bx}{a}-a^2$.

16. $\dfrac{6}{x-H}=\dfrac{5}{H-K}$.

17. $\dfrac{x-a}{x-3}=\dfrac{x-b}{x-4}$.

18. $\dfrac{3a-2x}{a-3x}=\dfrac{10x+a}{2a+15}$.

19. $\dfrac{5a}{6}\left(\dfrac{x}{a}+a\right)=\dfrac{3a}{4}\left(\dfrac{2x}{a}-a\right)-\dfrac{5}{12}$.

20. $\dfrac{a}{2}\left(\dfrac{3x}{a}-a\right)+3=\dfrac{7a}{8}\left(\dfrac{7x}{2a}+\dfrac{a}{2}\right)$.

21. $\dfrac{8x-5a}{3x}+\dfrac{13x-9a}{4x}=2$.

22. $\dfrac{12x+5a}{10}=\dfrac{6x+7a}{5}-\dfrac{18a^2}{5x}$.

23. $\dfrac{x-H}{x-2}-\dfrac{x-K}{x-1}=0$.

24. $xa+2b^2=a^2+ab+xb$.

25. $K(Hx+2H+K)=H^2(x-1)$.

Systems of Equations

Solve the following systems for x and y and check in both equations:

1. $3x+5y=19$.
   $5x-4y=7$.

2. $5x+3y=109$.
   $4y-5x=52$.

3. $11x+2y=23$.
   $9x-y=-26$.

4. $5x-4y=-43$.
   $2x+3y=15$.

5. $3x-y=12$.
   $2y+6x=12$.

6. $7x-3=4y$.
   $x+y-2=0$.

7. $6x - y = 0.9.$
$x + 2y = 0.8.$

8. $.75x + 1.5y = 3.$
$y - x = -16.$

9. $0.3x + 0.2y = 9.5.$
$0.2x + 0.3y = 10.5.$

10. $\frac{x-3}{5y} = -2.$
$x + 7y = 6.$

11. $\frac{5x}{3} + 2y = 17.$
$2x - \frac{4y}{3} = -2.$

12. $\frac{x}{2} + \frac{y}{3} = 1.$
$\frac{x}{4} - \frac{2y}{3} = 3.$

Radicals

Simplify:

1. $2\sqrt{3} - \frac{1}{2}\sqrt{12} + 4\sqrt{27}.$

2. $3\sqrt{14}\sqrt{21} - \frac{8\sqrt{12}}{\sqrt{2}}.$

3. $\frac{1}{2}\sqrt{27} + \frac{1}{2}\sqrt{147} - \sqrt{5\frac{1}{3}}.$

4. $\frac{8}{5}\sqrt{20} - \sqrt{\frac{1}{5}} - \sqrt{45}.$

5. $30\sqrt{\frac{1}{2}} - \frac{9}{2}\sqrt{8} + 9\sqrt{\frac{169}{2}}.$

6. $2\sqrt{108} - 6x\sqrt{5\frac{1}{3}} + \sqrt{3(8x-1)^2}.$

7. $6a\sqrt{\frac{2}{3}} - \frac{b}{3}\sqrt{54} + \sqrt{24b^2}.$

8. $\sqrt{18a} - 2\sqrt{\frac{a}{2}} + 3\sqrt{50a}.$

9. $\sqrt{16a^3 - 48a^2b} - \sqrt{9ab^2 - 27b^3}.$

Multiply the following and simplify the results:

10. $(2\sqrt{3} - 3\sqrt{5})$ by $(\sqrt{3} + 2\sqrt{5}).$

11. $(2\sqrt{3} - \sqrt{6})$ by $(\sqrt{3} + 3\sqrt{6}).$

12. $(2\sqrt{a} + 5\sqrt{b})$ by $(4\sqrt{b} - \sqrt{a}).$

13. $(3\sqrt{2} - 2\sqrt{3})$ by $(4\sqrt{2} + 3\sqrt{3}).$

14. $(3\sqrt{5} - 2\sqrt{10})$ by $(4\sqrt{5} + \sqrt{10}).$

15. $(a - \sqrt{ab} + b)$ by $(\sqrt{a} + \sqrt{b}).$

16. $(2\sqrt{6} + 5\sqrt{3} - 3\sqrt{2})$ by $(2\sqrt{3} - \sqrt{2}).$

17. $\sqrt[3]{24} \cdot \sqrt[3]{18}.$

Problems Involving Linear Equations. One or Two Unknowns

Number Problems

1. Find four consecutive numbers whose sum is 54.

2. Find four consecutive odd numbers whose sum is 56.

3. Find two numbers whose sum is 140 and whose difference is 8.

4. The difference between two numbers is 13 and their sum is 77. Find the two numbers.

5. The difference between two numbers is 12, and seven times the smaller number exceeds the greater by 30. Find the numbers.

6. A schoolroom contains 30 seats, some double and some single. If the seating capacity of the room is 42, how many of the seats are double?

7. Separate 24 into two parts such that one part shall be 3 less than twice the other part.

8. One machine can make 100 dozen buttons in 48 minutes; a smaller machine can make the same number in 1 hour and 12 minutes. How long will it take both machines together to make 100 dozen buttons?

9. Find three consecutive numbers such that if they are divided by 2, 3, and 4 respectively, the sum of the quotients will equal the next higher consecutive number.

10. The numerator and denominator of a certain fraction are in the ratio $2:3$; if 3 is subtracted from the numerator and 6 from the denominator, the value of the resulting fraction is $\frac{3}{4}$. What is the original fraction?

11. If 4 is added to both the numerator and the denominator of a certain fraction, the resulting fraction equals $\frac{3}{4}$; if 2 is subtracted from both numerator and denominator, the new fraction equals $\frac{1}{2}$. Find the original fraction.

12. A proper fraction is such that if its numerator is divided by 2 and its denominator is increased by 3, its value becomes $\frac{1}{5}$; if the fraction is multiplied by $\frac{4}{5}$, the difference of its terms is 53. Find the fraction.

13. The sum of two fractions whose numerators are 3 is three times the smaller fraction; three times the smaller subtracted from twice the larger is $\frac{3}{8}$. What are the fractions?

14. Divide \$1250 between two men so that the first man shall have \$270 more than the second man.

15. The difference between the squares of two consecutive numbers is 65. What are the numbers?

16. The difference between the squares of two consecutive even numbers is 156. Find the numbers.

Yosemite Falls

17. A baseball team has played 140 games during the season. It has won 38 games more than it has lost. How many has it won?

18. The height of the Zambezi Falls in Africa is 132 feet less than three times the height of Niagara Falls. The height of the Yosemite Falls is 300 feet less than five times the height of the Zambezi. The sum of their heights is 2024 feet. How high is each?

19. Twice the length of the Hoosac Tunnel in Massachusetts is 1320 feet more than the length of the St. Gothard Tunnel in Switzerland. If the combined lengths of these tunnels is 73,920 feet, what is the length of each?

20. The area of the state of Texas is 2728 square miles more than thirty-two times the area of New Jersey. If the total area of the two states is 274,120 square miles, what is the area of each state?

Percentage Problems

21. In building a macadam road the county pays 50% as much as the state, and the town pays 60% as much as the county; if the road through a town costs \$180,000, find the share of each.

22. Mr. Reynolds invested a part of \$20,000 in bonds paying 5% interest, and the rest in a mortgage paying 6%. His annual income from both investments was \$1060. How much had he invested in bonds? How much in the mortgage?

23. I have invested a part of \$12,000 in 4% bonds. If I invest the rest of it at 6%, I shall obtain the same annual income from both investments. How much have I invested in bonds?

Age Problems

24. Ella is two years older than Margaret. Ten years ago she was twice as old. How old is each now?

25. Frank is $\frac{2}{3}$ as old as Henry. After 8 years, he will be $\frac{3}{4}$ as old. How old is each now?

26. Four years ago I was half as old as my father. Six years hence I will be three fifths as old as he. How old am I now?

Coin Problems

27. How could you pay a bill of \$4.35 with nickels and dimes, using the same number of each?

28. A man has 19 coins, some of which are half-dollars and the rest are dimes. All together are worth \$7.90. How many are there of each?

29. A man has a certain number of dimes. He has 6 times as many quarters as the number-of-dimes-decreased-by-2. He has in all \$8.20. How many has he of each?

30. A boy has \$7.50 in quarters, dimes, and nickels. He has five times as many dimes as quarters, and three times as many nickels as dimes. How many has he of each?

Geometry Problems

31. A square grass plot would contain 73 square feet more if each side were one foot longer. Find the side of the plot.

32. The length of a room is 8 feet greater than its width; if each dimension is increased by 2 feet, the area will be increased by 60 square feet. Find the area of the floor.

33. The area of a certain rectangle is 2400 square feet; if its length is decreased 10 feet and its breadth increased 10 feet, its area will be increased 100 square feet. Find its length and breadth.

34. The length of the Pennsylvania Station in New York City is 80 feet less than twice the width. If the perimeter of the building is 2420 feet, what are the dimensions of the building?

35. A square grass plot would contain 71 square feet more if each side were 1 foot longer. Find the side of the plot.

36. A rectangle and a square are equal in area. The length of the rectangle is 6 feet more and its width 2 feet less than a side of the square. Find dimensions of each figure.

Work Problems

37. A father engaged his son to work on the condition that the son was to receive \$2 for each day he worked and to forfeit \$1 for each day he was idle; at the end of 20 days the son received \$34. How many days did he work?

38. Charles can wash his father's car in 3 hours, but his older brother John can do it in 2 hours. How long will it take both boys to wash the car if they work together?

39. Oscar can mow a lawn in 40 minutes. He and Donald together can mow it in 24 minutes. How long should it take Donald alone to mow it?

Mixture Problems

40. The water of Great Salt Lake contains about 20% of mineral substances. How much pure water must be added to 100 pounds of this water so that the mixture shall contain only 12% of minerals?

41. How much water must be added to a gallon of alcohol, which is 90% pure, to reduce it to 80% pure?

42. Three quarts of gasoline are mixed with 5 quarts of kerosene. How many quarts of kerosene must be added to make the mixture three fourths kerosene?

43. Milk is sold at 16¢ a quart and cream at 72¢ a quart. How many quarts of each will be needed to make 18 quarts of a mixture to sell for \$6.24?

Distance, Time, and Rate Problems

44. An automobile is traveling at the rate of 20 miles an hour. At what rate must a second automobile travel which starts 2 hours after the first and wishes to overtake it in 4 hours?

45. A is 60 miles from B. An automobile at A starts for B at the rate of 20 miles an hour at the same time that an automobile at B starts for A at the rate of 25 miles an hour. How long will it be before the automobiles will pass each other?

46. A man starts from a certain place and walks at the rate of $3\frac{1}{2}$ miles an hour; two and a half hours later another man starts from the same place and rides in the same direction at the rate of $8\frac{1}{2}$ miles an hour. In how many hours will the second man overtake the first?

47. A stream flows at the rate of 2 miles per hour; a launch can go at the rate of 8 miles per hour in still water. How far down the stream can the launch go and return if the complete trip can take only 6 hours?

48. A walks over a certain road at the rate of 3 miles an hour. Two hours after A leaves, B starts walking after A at the rate of 4 miles an hour. How many miles will A have gone when B overtakes him?

49. Two automobiles start at the same time from towns 330 miles apart and travel toward each other. If one averages 30 miles an hour and the other averages 25 miles per hour, in how many hours will they meet?

50. Two trains start from the same place and go away from each other. One runs at an average rate of 35 miles an hour and the other at an average rate of 40 miles an hour. In how many hours will they be 525 miles apart?

Word Problems

Drill in Algebraic Expression

1. In how many hours can a boy walk m miles at the rate of c miles per hour?

2. If a man's rate of rowing in still water is S miles an hour and the current flows at the rate of C miles an hour, express the man's rate (a) when rowing with the current, (b) when rowing against the current.

3. An odd number is represented by $2n + 1$. Represent the next two consecutive odd numbers.

4. Two trains leave a town at the same time and travel in the same direction; the faster one runs m miles per hour and the slower runs s miles per hour. In how many hours will they be k miles apart?

5. The age of the older of two sisters is now 6 years less than twice the age of the younger. If x represents the number of years in the age of the younger sister now, express by means of an equation the fact that 9 years ago the age of the older was two and one half times the age of the younger.

6. How many hours are there in d days?

7. If p pounds of tea cost c dollars, how many cents did the tea cost a pound?

8. If a man has x dollars and spends $\frac{1}{5}$ of that sum, how many dollars has he left?

9. If x represents the number of feet in the length of a line, express $\frac{1}{4}$ of it increased by 5 inches.

10. How many ounces are there in p pounds?

11. How many pounds are there in z ounces?

12. A merchant pays d dollars for n yards of velvet. What is the average price per yard?

13. How many pounds of sugar at b cents a pound would cost as much as e dozen of eggs at c cents a dozen?

14. How many yards are there in c inches?

15. A boy will be y years old 2 years hence. How old was he 4 years ago?

16. A book cost d dollars. How many dollars will have to be paid for a second book if it costs c cents more than the first?

17. A rectangle is b inches long and c inches wide. Find its area and its perimeter.

18. The width of a rectangle is x yards and its length is y yards. Express its area and its perimeter.

19. What will be the amount (simple interest) at the end of 5 years if P dollars are invested at $r\%$?

20. If $2n + 1$ is an odd number, express the next two higher even numbers.

21. (a) The three digits of a number are h, t, and u. Represent the number. Represent the number with its digits reversed.

(b) A man earns e dollars a month and spends s dollars a month. How many dollars will he save in 3 years?

22. John is y years old now. Write the equation showing that four times his age three years ago equals twice his age five years hence.

23. From a board b feet in length a piece c feet long is cut from one end and a piece d inches long is cut from the other end. How many inches in length remain?

Quadratic Equations

Solve for x and check:

1. $2x^2 + x - 21 = 0$.

2. $10x^2 + 3ax - a^2 = 0$.

3. $\dfrac{7x}{6} = \dfrac{1}{2} - x^2$.

4. $\dfrac{x}{3(x-1)} = \dfrac{x-2}{2}$.

5. $\dfrac{3x}{2} = x + \dfrac{x+7}{x-3}$.

6. Solve $x^2 - 21 = 2x$ and find the values of x to the nearest tenth:

7. Find to the nearest tenth the roots of

$$5 - \frac{x^2}{6} = \frac{4x}{3}.$$

8. Find the roots of the following equation correct to the nearest tenth: $x^2 + 2x = \frac{3}{4}$.

9. Solve for c correctly to two decimal places:

$$c^2 + 4c - 1 = 0.$$

10. By completing the square solve the equation

$$2x^2 - 7x + 4 = 0$$

and find the roots correct to two decimal places.

Quadratic Equations. Problems

1. Find the number whose square diminished by 20 is equal to 8 times the number.

2. One number is $\frac{2}{3}$ of another number, and their product plus their sum is 69. Find the numbers.

3. A rectangular building lot is known to contain 2700 square feet; one side is 15 feet longer than the other. Find the dimensions of the lot.

4. The length of a given rectangle exceeds its width by 4 yards and the area of the rectangle is 82 square yards. Find the dimensions of the rectangle.

5. A picture 8 inches by 12 inches is placed in a frame of uniform width. If the area of the frame equals the area of the picture, find the width of the frame. (Draw a diagram.)

6. A rectangular piece of paper is twice as long as a square piece and 3 inches wider; the area of the rectangular piece is 108 square inches. Find the dimensions of the square piece.

7. The frame for a square picture is 2 inches wide and the area for the opening for the picture is 54 square inches.

Find the outside dimensions of the frame and state whether or not both results obtained are admissible.

8. An automobile goes to a place 72 miles distant and then returns, the round trip occupying 9 hours. The average rate of speed in returning is 12 miles per hour faster than the rate in going. Find the rate of speed in (*a*) going, (*b*) returning.

Formulas

1. Solve for F the formula $C = \frac{5}{9}(F - 32)$.

2. Solve for l the formula $S = \frac{n}{2}(a + l)$.

3. In the formula $h^2 = a^2 + b^2$, find the value of b to two decimal places, *i.e.* to the nearest hundredth, when $h = 7$ and $a = 1$. $h = 14$, $a = 2$. How does b change when h and a are doubled?

4. (a) Evaluate the formula $\frac{1}{t}\left(\frac{T^2}{R^2} - 1\right)$ when $T = 8$, $R = 2$, $t = 10$.

(b) Solve for M the formula $L = \frac{Mt - g}{t}$.

5. Solve for v in terms of h and g:

$$h = \frac{v^2}{2g}.$$

What is the value of v if $h = 25$ and $g = 32$?

6. Eliminate a in the formulas (equations) $F = Ma$ and $s = \frac{1}{2}at^2$, obtaining a formula that does not contain a.

7. The area of a trapezoid is expressed by the formula $x = \frac{h}{2}(b + b')$, in which h denotes the height, b and b' the two bases.

(a) Find h in terms of a, b, and b'.

(b) Find the value of b, if $a = 35$, $b' = 10$, and $h = 5$.

8. Find the value of $\sqrt{b^2 - 4ac}$ when

$a = 2, \quad b = 3, \quad c = -5,$
$a = 8, \quad b = -8, \quad c = 2,$
$a = \frac{1}{8}, \quad b = \frac{1}{2}, \quad c = \frac{1}{4}.$

9. Find the value of $-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ when

$a = 1, \quad b = -7, \quad c = 12,$
$a = 6, \quad b = 1, \quad c = -1,$
$a = 3, \quad b = -6, \quad c = 10.$

10. In the formula $R = \frac{gs}{g + s}$:

(a) Solve for s in terms of R and g.

(b) Find the value of s to the nearest hundredth when $R = 2.75$ and $g = 3.9$.

11. In the formula $T = 2\pi R(R + H)$:

(a) Solve for H in terms of the other letters.

(b) Find the value of H to the nearest tenth if $\pi = 3.14$, $R = 10$, and $T = 794.42$.

12. In the formula $S = \frac{a}{2}(2t - 1)$:

(a) Find the value of t in terms of a and S.

(b) Find the value of t when $a = 5.65$ and $S = 73.45$.

13. If the list price of an article is represented by L, and the discount a merchant offers from the list price is represented by $d\%$, how would you represent the selling price in terms of L and d? Representing the selling price by S, make a formula for the selling price.

14. (a) Solve for r: $C = \frac{E}{R + r}$.

(b) Find the value of r to the nearest tenth when $C = 15.6$, $R = 4.3$, and $E = 110.7$.

15. If the length of a room is L yards, the width W yards, and the height H yards, write in terms of these letters an expression to represent the total surface of (1) the four walls, (2) the ceiling.

16. If C represents the cost of an article and r the gain per cent, express the selling price in terms of C and r.

Formula Making

Make up formulas from the following tables:

Observe how one letter changes as the other changes.

1.

n	0	1	2	3	4	5
m	0	1	8	27	64	125

2.

x	1	2	3	4	5	6
y	0	3	8	15	24	35

3.

x	0	1	2	3	4	5	6
y	0	2	8	18	32	50	72

4.

x	0	1	3	4	5	6	7	8
y	0	2	12	20	30	42	56	72

5.

x	0	1	2	3	4	5	6
y	-2	-1	2	7	14	23	34

6.

n	0	1	2	3	4	5	6
c	1	2	5	10	17	26	37

7.

m	1	2	3	4	5	6
n	$\frac{1}{2}$	2	$4\frac{1}{2}$	8	$12\frac{1}{2}$	18

8.

x	1	2	4	5	6	7	8
y	$1\frac{1}{2}$	3	6	$7\frac{1}{2}$	9	$10\frac{1}{2}$	12

9.

a	1	2	2	4	1	2	6	2
b	1	2	3	3	6	5	3	4
c	2	8	13	25	37	29	45	20

Miscellaneous Exercises

1. Solve for a, $A = \frac{a(b+c)}{2}$.

2. Show that $x^3 = x(x-1)(x-2) + 3x(x-1) + x$.

3. What is the dividend if the divisor is $x^2 - x + 1$, the quotient is $x^2 + x + 1$, and the remainder is $x + 1$?

4. Prove that

$$xy - 3(x-1)(y-1) + 3(x-2)(y-2) - (x-3)(y-3) = 0.$$

5. Simplify:

$$x - [x + y - \{x + y + z - (x + y + z + w)\}].$$

6. Show that

$$(a+3)(b+3) - 3(a+1)(b+1) + 3(a-1)(b-1) - (a-3)(b-3) = 0.$$

7. What is the value of $xyz(xy + yz + zw + xw) \div (x+y)(x+z)(x+w)$ when $x = 1$, $y = 3$, $z = -5$, and $w = 0$?

8. A square grass plot would contain 93 square feet less if each side were 3 feet shorter. How long is a side of the plot? What is its area?

9. Solve the equation $(x - 3)(x + 4) = x^2 + 6$.

10. Find two numbers such that twice their difference is less by two than the larger number and greater by one than the smaller number.

11. Find the continued product of $x - 1$, $x + 1$, $x^2 + 1$, and $x^4 + 1$.

12. Find the H. C. F. of $3x^2 - 8x - 3$ and $2x^2 - 3x - 9$.

13. Prove that

$$x^2 + 3(x - 2y)^2 = 3(x - y)^2 + (x - 3y)^2.$$

14. Prove that the product of any two numbers is equal to one fourth the difference of the square of their sum and the square of their difference.

15. Simplify: $\dfrac{(x^2 + 14x + 49)(x - 1)}{x^2 + 6x - 7}$.

16. Find the L. C. M. of $x^2 + 1$, $(x - 1)^2$, $(x + 1)^2$, and $x^2 - 3x + 2$.

17. Write the sum of the roots of each of the following equations:

(a) $x^2 - 4b^2 = 0$. (c) $x^2 - 6x + 8 = 0$.

(b) $x^2 + 3x - 4 = 0$.

18. Find the sum of the squares of the roots of the equation $x^2 - 16x + 48 = 0$.

19. What number is such that its square exceeds itself by 56?

20. Subtract $ax^2 - 4$ from zero, and add the difference to the sum of $2x^3 - 5x$ and unity.

21. Prove that

$$x^2 - xy + y^2 - \frac{2y^3}{x + y} = \frac{x^3 - y^3}{x + y}.$$

22. Henry rode his bicycle 8 miles an hour for two hours, rested 1 hour and then rode for 1 hour at 6 miles an hour. What uniform rate would have enabled him to reach his destination in the same time? Solve by a graph.

23. Simplify $2a - 3(a - 1) - [a - 2(2a - 1)]$.

24. Change the fraction $\frac{m - n}{m + n}$ to an equivalent fraction having $m^2 - n^2$ for a denominator.

25. Solve $\frac{a - bx}{c} + b = \frac{bc - x}{c}$.

26. Factor $x^4 + 2x^2y^2 + y^4 - a^2b^2$.

27. Find the H. C. F. and the L. C. M. of

$$2x^2 + xy - y^2 \text{ and } 6x^2 + xy - 2y^2.$$

28. Solve $\frac{cx - d}{dx} + \frac{dx + c}{cx} = 2 + \frac{c - d}{cdx}$ for x.

29. What must be the value of m if $x - 5$ is a factor of $2x^2 - mx - 35$?

30. What must be the value of k if 2 is a root of

$$3x^2 - 12x + k = 0?$$

31. If $a = \frac{2b - 3}{b^2}$ and $c = \frac{a + 1}{a + 3}$, find c when $b = -1$.

32. Simplify and arrange in ascending powers of m:

$$(m - 2)^4 - (m - 1)^3 + 3(2 - m)^2.$$

33. If $x = 4$ and $y = 5$, find the value of

$$x^2 - 3(x - y) + 4(x - 2y).$$

34. From the sum of $h - k + r - x$ and $2h - k + 2r - x$ subtract $3h + 2k + 2r - 3x$.

35. Solve $5x - (2x + 3) = 12$. Check.

36. Reduce to lowest terms $\frac{n^2 - 7n + 12}{n^2 - 2n - 3}$.

37. Give an example of (a) a linear equation, (b) a radical expression, (c) a trinomial, (d) a quadratic equation, (e) simultaneous equations, (f) an exponent.

38. The admission tickets for an entertainment were 25¢ each for adults and 10¢ each for children; the turnstile shows that 358 persons entered and the gate receipts were \$62.65. How many children entered?

39. Give examples of

(a) Factor, (b) index, (c) reciprocal, (d) binomial, (e) numerical equation, (f) degree of a term.

40. A man wishing to give 35 cents to each of a certain number of boys, finds that he lacks 20 cents, but that he can give each boy 33 cents and have 4 cents left. What sum has he and how many boys are there?

41. A number of boys go camping, agreeing to divide the expenses among themselves. If the number had been two more, each would have paid \$12 less; if the number had been three less, each would have paid \$24 more. How many boys were there and how much did each boy pay?

42. When the values of x and y are very small $\dfrac{1+x}{1+y}$ may be taken as nearly equal to $1 + x - y$. Find to five decimal places the error in this statement when $x = 0.02$ and $y = 0.03$.

43. Solve for x and y and check:

$$\frac{x}{a} + \frac{y}{b} = 2.$$

$$bx - ay = 0.$$

44. Solve for x and y and group your answers:

$$x(y+1) = 4.$$

$$3x = 2y.$$

45. In a purse containing \$2.50 there are one half as many quarters as 5-cent pieces and one third as many dimes as 5-cent pieces. How many pieces of each kind are there?

46. Reduce to lowest terms: $\dfrac{3c^2 - 6c}{6c^2 - 24}$.

47. Solve for x and y:

$$ax + by = 2.$$
$$abx + aby = a + b.$$

48. Give the name applied to the 3 in each of the following and explain its meaning in each case:

$$3a,\ a^3,\ \sqrt[3]{a},\ \frac{a}{3}.$$

49. Write in symbols: The square of twice a number diminished by twice the square of the same number.

50. If the length of a rectangle is represented by l feet, represent the length of a rectangle: (1) 5 feet shorter, (2) 5 feet longer, (3) 5 times as long, (4) one fifth as long.

51. In sending a telegram there is a fixed rate for the first 10 words and a fixed rate for each additional word. If a message of 31 words costs 98 cents and a message of 45 words costs \$1.40, what are these two fixed rates?

52. Perform the indicated operations:

$$(2a - 3)^2 - 3a(a - 2) - (3 - a)^2.$$

53. Solve $\begin{cases} 9y^2 - 6y - 5 = 3x. \\ 9y + x + 5 = 0. \end{cases}$

54. Solve for x and y, correctly group your answers and check:

$$\begin{cases} 2x - 3y = 3. \\ x^2 - 4xy = -3. \end{cases}$$

55. Solve $\begin{cases} xy = 6. \\ 3x - 2y = 16. \end{cases}$

56. Solve $\begin{cases} x^2 - xy + y^2 = 7. \\ 2x - 3y = 0. \end{cases}$ Check one set of roots.

57. Solve for x and y and check:

$$x^2 + y^2 = 26.$$
$$x - y = -4.$$

58. Solve the equations

$$4x + 5y = 6.$$
$$xy = -2.$$

Check the solutions.

59. Solve for x and y and group your answers:

$$x^2 - 3y^2 = 13.$$
$$1 - x + 2y = 0.$$

60. Solve the following for x and y, correctly group your answers, and check either set:

$$\begin{cases} 3xy - 10x = y. \\ 2 - y = -x. \end{cases}$$

61. Solve for x and y, group your answers, and check one set.

$$3x + 2y = 5.$$
$$xy + 3 = 6x.$$

General Review

1. In a lever $\frac{w}{w'} = \frac{l'}{l}$ where w and l and w' and l' are corresponding values of the length and width. Are w and l directly or inversely proportional?

2. Construct with your protractor an angle of 45 degrees in a right triangle. Measure the sides of the triangle and find the value of tan 45°.

3. Is the positive square root of $\frac{1}{4}$ greater or less than $\frac{1}{4}$?

4. In terms of n represent three numbers which have the ratio 3 : 5 : 8. If the sum of these numbers is 64, find the numbers.

5. Does $(a^3)^2 = a^5$ or a^6?

6. If the area of a square is represented by $a^2 - 6ab + 9b^2$, what should represent one of its sides?

7. Prove that $\sqrt{a^2 - \frac{a^2}{4}} = \frac{a}{2}\sqrt{3}$.

8. Solve by a graph the system of equations

$$\begin{aligned} x + y &= 7. \\ 2x - 3y &= -1. \end{aligned}$$

Check algebraically.

9. Which of the following statements is true and which is false?

$$\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}.$$

$$\frac{m+n}{a} = \frac{m}{a} + \frac{n}{a}.$$

10. If $3x = 4y$, what is the ratio of x to y?

Hint: Divide both members by $3y$.

11. Construct a graph of the simple interest formula $I = prt$ if $r = 6\%$ and $t = 5$ years. Use values of p, from \$100 to \$1000. Even \$100's.

12. In example 11, what change will take place in I if p is doubled? Halved? Trebled?

13. The area of a rectangle is 24. Make a table of possible values of l and w and show that l varies inversely as w.

14. Draw a graph of the inverse variation $lw = 24$.

15. What change takes place in the sine of an acute angle if the angle is increased? In the tangent? In the cosine?

16. By reference to your tables of trigonometric functions show that $\frac{\sin 30^\circ}{\cos 30^\circ} = \tan 30^\circ$.

17. In $y = \frac{10}{10 - x}$ as x varies from 0 to 9, y varies from . . . to . . .

18. Solve the system $\left.\begin{aligned} x - 2y &= 6 \\ x^2 + y^2 &= 65 \end{aligned}\right\}$ by substitution.

19. In order that there may be 4 fewer telephone poles to the mile it will be necessary to set them 12 feet farther apart. How far apart are they now?

20. Is $\dfrac{x^2 + x}{x + 1} = 5$ a quadratic equation? Why?

21. Solve, obtaining values to the nearest tenth:

$$.3\,x^2 + .8\,x = .17$$

22. Classify as true or false: x and y positive.

(a) $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.

(b) $\sqrt{x} \cdot \sqrt{y} = \sqrt{xy}$.

(c) $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$.

23. If an angle is less than 45°, which is greater, $\sin x$ or $\cos x$?

24. Graph the quadratic function $2\,x^2 + x - 1$.

25. $V = 50 + 10\,t^2$. Solve this formula for t.

26. Complete the table and write a formula showing how y depends on x. How does y change as x decreases?

x	0	1	2	3	4	5	6	7
y	2	3	4	5				

27. By using the table, test the truth of the formula

$$\sin^2 A + \cos^2 A = 1 \text{ if } A = 60°.$$

Hint: $\sin^2 A = (\sin A)^2$.

28. Solve $\sin^2 A + \cos^2 A = 1$ for $\cos A$. Find $\cos A$ if $\sin A = 0, \frac{1}{2}, \frac{1}{2}\sqrt{3}, 1$.

29. Solve the system $\left.\begin{aligned} xy &= 6 \\ x - y &= 1 \end{aligned}\right\}$ by substitution, and check.

30. $y = \dfrac{3\,x + 2}{5}$. Here y is a function of x. Express x as a function of y.

31. If 3 tractors can harvest 28 acres in a day, how many tractors will be needed to harvest 84 acres in a day at the same rate? Solve by proportion.

TRACTORS AT WORK

32. Simplify $\left(\frac{a}{b}+1\right)\div\left(\frac{a}{b}-1\right)$.

33. If x represents the left digit and y the right digit of a number, what does $10x+y$ represent?

34. Solve and check $x^2-\frac{6}{a^2}=-\frac{5x}{a}$ for x.

35. The formula for the volume of a sphere is $V=\frac{1}{6}\pi d^3$. If $\pi=3.14$ and $d=1.2$, find the volume to the nearest tenth.

36. $h=\sqrt{a^2+b^2}$. Complete the following table.

a	3	12	5	24	9	15	6	12	4
b	4	9	12	10	40	8	8	16	$4\sqrt{3}$
h									

How does h change if a and b are doubled?

37. Extract the square root of 20.45 to the nearest tenth.

38. Using factoring, find in a short way the value of $19^2 - 11^2$.

39. Reduce to lowest terms $\dfrac{x^2 - 4y^2}{x^2 - 5xy + 6y^2}$.

40. Simplify $\dfrac{2xy}{x^2 - y^2} + \dfrac{x}{x + y}$.

41. Draw a graph of the formula $F = \frac{9}{5}C + 32$.

42. $t = \dfrac{d}{r}$. If $d = 500$, draw a graph of this formula. When r is increased, what change takes place in t?

43. A merchant wishes to mix walnuts worth 40 cents a pound with pecans worth 50 cents a pound so as to have a mixture of 60 pounds which he can sell at 43 cents a pound. How many pounds of each must he take?

44. Why should $2n$ represent an even rather than an odd integer?

45. A father is now 8 times as old as his son. In 3 years he will be only 5 times as old. How old is each now?

46. If $I = prt$ and p and r are constant, then I varies as . . . If p is constant, then I varies as . . . and . . .

47. In the formula $A = lw$ what happens to A if w is doubled while l remains constant? If l is doubled while w remains constant? If both l and w are doubled? If l is doubled while w is halved?

48. How long must a ladder be to reach a window 24 feet from the ground if it is set so that it makes an angle of 22° with the building?

49. Draw a graph of the equation $y = \sin x$, using the table, and considering values of x at intervals of 10° from 0° to 90°.

EXAMINATIONS

Elementary Algebra

Part I

Answer all questions in this part. Each question has $2\frac{1}{2}$ *credits assigned to it; no partial credit should be allowed. Each answer must be reduced to its simplest form.*

1. What algebraic expression must be added to $3x^2 - 5x + 6$ to make the result 0? *Ans.* .....

2. Write the expression $a^2 - 7b + 9$ in an equivalent form in which the last two terms are inclosed in a parenthesis preceded by a *minus* sign. *Ans.* .....

3. In the formula $V = \frac{1}{3}hb$, find V when $h = 10$ and $b = 28$. *Ans.* .....

4. In the formula $h = \dfrac{v}{lw}$, express l in terms of v, w, and h.

Ans. .....

5. A dealer sold n automobile tires at s dollars each and thereby gained r dollars; what was the cost of one tire?

Ans. .....

6. Factor $12a^2 - 27b^2$. *Ans.* .....

7. If $x = 3y + 2$, express x^2 as a trinomial in y.

Ans. .....

8. Solve for x:

$$\frac{7x}{12} - \frac{1}{4} = 2x - \frac{5}{3}.$$ *Ans.* .....

9. In the following set of equations find the value of a in terms of c and d:

$$2a - b = c$$
$$a + 3b = d.$$

Ans. .....

10. Simplify $\sqrt{3a^2} \times \sqrt{18b^2}$ *Ans.* .....

11. In the equation $x^2 + 2x - 15 = 0$, which of the two numbers -3 and -5 is a root; that is, which number satisfies the equation? *Ans.* .....

12. Solve for x:

$$2x^2 + 5x = 3.$$

Ans. .....

13. If the area of a square is 14, find a side correct to the *nearest tenth.* *Ans.* .....

14. If $4 : y = 8 : 10$, find y. *Ans.* .....

15. If $y = -12x$ and x is positive, does y increase or decrease as x increases? *Ans.* .....

16. What number added to 6% of itself equals 31.8?

Ans. .....

17. Using $-20m$ as the second term, write a trinomial that is a perfect square. *Ans.* .....

18. Which of the following pairs of values for x and y satisfies the equation $2x - y = 6$?

$x = 4, y = -2.$ $x = 4, y = 2.$
$x = 2, y = 2.$ $x = 0, y = 6.$

Ans. .....

19. An airplane starting from Albany flies a distance d miles in 3 hours. It takes a second plane 2 hours longer to fly the same distance. Write an equation stating that the difference between the rates of the two planes is 25 miles an hour. *Ans.* .....

20. The temperature readings at a government station on a certain winter afternoon were as follows:

2 P.M. ..... $+ 10°$
4 P.M. ..... $+ 14°$
6 P.M. ..... $+ 8°$

Represent this information on a diagram by points. Use the second line from the bottom as the horizontal axis, and plot time on this axis; use the second line from the left as the vertical axis, and plot temperature on this axis.

Part II

Answer five questions from this part. Full credit will not be granted unless all operations (except mental ones) necessary to find results are given; simply indicating the operations is not sufficient. Each answer should be reduced to its simplest form.

21. Solve for x:

$$\frac{c - x}{nx} - \frac{n}{cx} = -\frac{1}{c}.$$

22. A father is now 24 years older than his son; in 8 years the father will be twice as old as the son. Find the present age of each.

23. The distance between two villages, L and M, is 8 miles less than the distance from M to a third village S. One half the distance between M and S is 6 miles less than the distance between L and M. Find the number of miles from L to M.

24. Four times the square of a certain positive number exceeds 8 times the number by 12. Form the equation, solve, and check.

25. *a.* Write a formula for the following rule:

To find the approximate number of bushels (n) in a bin, multiply the length (l) by the width (w) by the height (h), each expressed in feet, and divide this product by 1.25.

b. In the formula $i = prt$, used in computing simple interest, i represents the interest, p the principal, r the rate, and t the time in years. Using this formula, compute the interest on \$600 for 2 years at 5%.

26. Indicate whether *each* of the following statements is true or false: [Write the letters *a*, *b*, *c*, *d*, *e* in a column and then write the word *true* or *false* after each letter.]

a. One root of the equation $x^2 - 5x + 6 = 0$ is 3.

b. $$\frac{-6}{2+x} = \frac{6}{2-x}.$$

c. $$\frac{a-b}{a+b} \times \frac{a^2-b^2}{a^2-2ab+b^2} = 1.$$

d. If a and b are positive integers, the square of their sum is greater than the sum of their squares plus their product.

e. In the formula $A = \pi r^2$, if r is doubled, then A is doubled.

27. Copy and complete *each* of the following statements:

a. In clearing a fractional equation of fractions we ..... each member of the equation by the least common denominator.

b. $$\frac{3m^2 - 7mr - 6r^2}{m - 3r} = \text{.....}$$

c. In the equation $ax + b^2 = bx + a^2$, $x =$ .....

d. If the width of a rectangular field is 8 rods less than its length (l), its perimeter in terms of l is ..... rods.

e. In the set of equations $x^2 + y^2 = 13$ and $x + y = 5$, there are ..... pairs of values for x and y that satisfy the equations.

28. $A = s^2$ is the formula for the area of a square.

a. Make a table showing the values of A when s equals 1, 2, 3, 4.

b. Plot a graph showing these relations.

c. Is the graph a broken line or a curved line?

Elementary Algebra

Part I

Answer all questions in this part. Each question has $2\frac{1}{2}$ *credits assigned to it; no partial credit should be allowed. Each answer must be reduced to its simplest form.*

1. A customer purchases n pounds of sugar at c cents a pound and gives in payment a one-dollar bill. Express in cents the change he should receive. *Ans.* .....

2. From the sum of $2a - 10b$ and $3a + 14b$ subtract $3a - 2b$. *Ans.* .....

3. In the formula $2\pi r = C$, if $C = 88$ and $\pi = \frac{22}{7}$, find r. *Ans.* .....

4. Solve the following equation for y:

$$5y - 2(y - 5) = 17.$$ *Ans.* .....

5. Solve the following set of equations for x:

$$x - 3y = 8$$
$$2x + 3y = 7.$$ *Ans.* .....

6. Solve the following equation for x in terms of a and b:

$$ax - a^2 = bx - b^2. \quad \textit{Ans.} \ldots\ldots$$

7. The formula $B = 2A$ is obtained by inspection from the table at the right.

A	1	2	3	...
B	2	4	6	...

What formula is obtained from the table given at the right of this question? *Ans.* .....

A	1	2	3	...
B	3	6	9	...

8. Factor $P + Prt$. *Ans.* .....

9. Solve the following equation for d:

$$\frac{d}{2} - \frac{d}{5} = 6. \quad \textit{Ans.} \ldots\ldots$$

10. Express as a single fraction in its lowest terms

$$\frac{a+b}{2} - \frac{2a-b}{6}. \quad \textit{Ans.} \ldots\ldots$$

11. Express as a single term

$$2\sqrt{32} - 3\sqrt{2}. \quad \textit{Ans.} \ldots\ldots$$

12. The parcel-post rate for a certain zone is expressed by the formula $c = 2w + 6$. Solve this formula for w.

Ans. .....

13. If x and y are positive and $y = \frac{3}{x}$, does y increase or decrease as x increases? *Ans.* .....

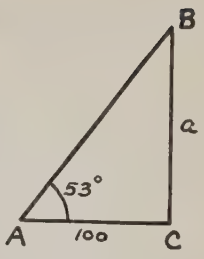
14. Find $\sqrt{68}$ to the *nearest tenth*. *Ans.* .....

15. Simplify $(-\frac{2}{5}ab^2)^2$. *Ans.* .....

16. In a certain freshman class there are a boys and b girls. Express the ratio of the number of girls to the total number of pupils. *Ans.* .....

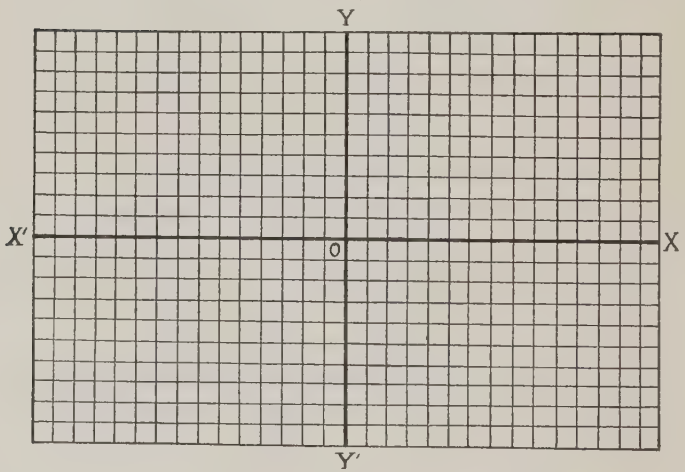
17. The cosine of a certain acute angle is .8712. Find the angle to the *nearest degree.* *Ans.* .....

18. Using the tables, find to the *nearest tenth* the value of a in the right triangle ABC. *Ans.* .....



19–20. *a.* On the diagram below represent by a graph the equation $y = 2x$.

b. On the graph plotted in answer to *a*, mark with the letter P the point for which $x = 4$.



Part II

Answer five questions from this part. Full credit will not be granted unless all operations (except mental ones) necessary to find results are given; simply indicating the operations is not sufficient. Each answer should be reduced to its simplest form.

21. Solve the following set of equations for x and y and check:

$$2x + 3y = 6$$
$$3x + 5y = 15.$$

22. In a foul-shooting contest the points made by Howard and Kenneth were in the ratio 8 : 11. Kenneth made 12 more points than Howard. Find the number of points made by each.

23. If 3 is subtracted from the numerator of a certain fraction, the value becomes $\frac{2}{3}$. If 10 is added to the denominator of the original fraction, the value becomes $\frac{1}{2}$. Find the original fraction.

24. If one side of a square is increased by 2 and an adjacent side is decreased by 2, the area of the resulting rectangle is 32. Find one side of the square.

25. Indicate whether *each* of the following is true or false: [Write the letters *a* to *j* in a column and then write the word *true* or *false* after each letter.]

a. $3\,a^2 \times 2\,a^3 = 6\,a^6$.

b. If $\dfrac{2}{x} = 6$, then $x = 3$.

c. If $A = 45°$, then $\sin A = \cos A$.

d. If $ax = b$, then $x = b - a$.

e. The common factor of $a^2 - b^2$ and $a^2 - ab$ is $a - b$.

f. If an even integer is represented by $2\,n$, then the next larger even integer is represented by $2\,n + 2$.

g. $\dfrac{ab + c}{a} = b + c$.

h. $\sqrt{a^2 + b^2} = a + b$.

i. $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$.

j. $\dfrac{-a}{a - 1} = \dfrac{a}{a + 1}$.

26. $K = \frac{1}{2} h(b_1 + b_2)$ is the formula for the area of a trapezoid, h representing the altitude and b_1 and b_2 the bases If $K = 252$, $b_1 = 18$, and $b_2 = 24$, find h.

27. A boy selling newspapers earns on 5 consecutive days the following amounts: 10¢, 3¢, 17¢, 25¢, 20¢.

a. Represent this information by a bar graph.

b. Represent by a sixth bar his average daily earnings.

The following question is based on one of the optional topics in the syllabus and may be substituted for any other question in part II.

28. Solve for x to the *nearest tenth:*

$$x^2 - 2x = 23.$$

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$$\begin{array}{r} 4. \\ \sqrt{25.0} \\ 16 \\ \hline 4 \quad 6.0 \end{array}$$